PART A

Unit - II

Fourier Transforms

2.1 Introduction

Many engineering problems lead to ordinary or partial differential equations which have to be solved under various types of conditions formulated from the problem. We are already familiar with the solution of higher order ordinary differential equations with initial conditions (*initial value problems*) using Laplace transforms. Solution of some partial differential equations with boundary conditions (*boundary value problems*) can be obtained with the help of Fourier transforms.

2.2 Infinite Fourier transform (Complex Fourier transform) and inverse Fourier transform

The *infinite Fourier transform* or simply the *Fourier transform* of a real valued function $f(x)$ is defined by

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx \qquad \dots (1)$$

provided the integral exists. On integration we obtain a function of u which is usually denoted by $F(u)$ or $\hat{f}(u)$

The inverse Fourier transform of $F(u)$ denoted by $F^{-1}[F(u)]$ or $F^{-1}\left[\hat{f}(u)\right]$ is defined by the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du \qquad \dots (2)$$

On integration we obtain a function of x. That is

$$f(x) = F^{-1}[F(u)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du$$

Note : **(1)** The definitions are deduced from the Fourier integral

$$f(x) = \frac{1}{2\pi}\int_{u=-\infty}^{\infty}\ \int_{t=-\infty}^{\infty} f(t)\, e^{iu(t-x)}\, dt\, du$$

(2) In view of the term e^{iux} present in the definition of the Fourier transform, it is also called the ***Complex Fourier transform.***

2.3 Properties of Fourier Transform

1. *Linearity property*

If $c_1, c_2, \cdots c_n$ *are constants then*

$$F[c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)]$$
$$= c_1 F[f_1(x)] + c_2 F[f_2(x)] + \cdots + c_n F[f_n(x)]$$

Proof : By the definition,

$$F[c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)]$$

$$= \int_{-\infty}^{\infty} [c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)] e^{iux} dx$$

$$= c_1 \int_{-\infty}^{\infty} f_1(x) e^{iux} dx + c_2 \int_{-\infty}^{\infty} f_2(x) e^{iux} dx + \cdots + c_n \int_{-\infty}^{\infty} f_n(x) e^{iux} dx$$

$$= c_1 F[f_1(x)] + c_2 F[f_2(x)] + \cdots + c_n F[f_n(x)]$$

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2. *Change of scale property*

If $F[f(x)] = \hat{f}(u)$, *then* $F[f(ax)] = \frac{1}{a}\hat{f}\left(\frac{u}{a}\right)$

Proof : By the definition,

$$F[f(ax)] = \int_{-\infty}^{\infty} f(ax) e^{iux} dx$$

Put $ax = t$ $\therefore$ $dx = dt/a$ and t also varies from $-\infty$ to ∞

Now $F[f(ax)] = \int_{-\infty}^{\infty} f(t) e^{iut/a} \frac{dt}{a}$

ie., $F[f(ax)] = \frac{1}{a} \int_{-\infty}^{\infty} f(t) e^{i\frac{u}{a}t} dt = \frac{1}{a}\hat{f}\left(\frac{u}{a}\right)$

Thus $\boldsymbol{F[f(ax)] = \frac{1}{a}\hat{f}\left(\frac{u}{a}\right)}$

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3. ***Shifting property***

If $F[f(x)] = \hat{f}(u)$ *then* $F[f(x-a)] = e^{iua}\hat{f}(u)$

Proof : By the definition we have,

$$\hat{f}(u) = F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{iux}dx$$

Hence $F[f(x-a)] = \int_{-\infty}^{\infty} f(x-a)e^{iux}dx$

Put $x-a = t$ $\therefore$ $dx = dt$, t also varies from $-\infty$ to ∞

Now $F[f(x-a)] = \int_{-\infty}^{\infty} f(t)e^{iu(t+a)}dt = e^{iua}\int_{-\infty}^{\infty} f(t)e^{iut}dt = e^{iua}\hat{f}(u)$

Thus $\boldsymbol{F[f(x-a)] = e^{iua}\hat{f}(u)}$

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4. ***Modulation property***

If $F[f(x)] = \hat{f}(u)$ *then* $F[f(x)\cos ax] = \frac{1}{2}\left[\hat{f}(u+a)+\hat{f}(u-a)\right]$

Proof : By the definition

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{iux}dx$$

$$\therefore \quad F[f(x)\cos ax] = \int_{-\infty}^{\infty} f(x)\cos ax\, e^{iux}dx$$

ie.,

$$= \int_{-\infty}^{\infty} f(x)\cdot\frac{e^{iax}+e^{-iax}}{2}\cdot e^{iux}dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(x)e^{i(u+a)x}dx + \int_{-\infty}^{\infty} f(x)e^{i(u-a)x}dx\right]$$

$$= \frac{1}{2}\left[\hat{f}(u+a)+\hat{f}(u-a)\right]$$

Thus $\boldsymbol{F[f(x)\cos ax] = \frac{1}{2}\left[\hat{f}(u+a)+\hat{f}(u-a)\right]}$

2.4 Fourier cosine and Fourier sine transforms Inverse Fourier cosine and Inverse Fourier sine transforms

If $f(x)$ is defined for all positive values of x, we define the following

$$F_c[f(x)] = \int_0^\infty f(x)\cos ux\,dx = F_c(u)\cdots \qquad \textbf{\textit{Fourier cosine transform}}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(u)\cos ux\,du\cdots \qquad \textbf{\textit{Inverse Fourier cosine transform}}$$

$$F_s[f(x)] = \int_0^\infty f(x)\sin ux\,dx = F_s(u)\cdots \qquad \textbf{\textit{Fourier sine transform}}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(u)\sin ux\,du\cdots \qquad \textbf{\textit{Inverse Fourier sine transform}}$$

Note : *The following properties concerning Fourier cosine and Fourier sine transforms can easily be established as in the case of Fourier transform.*

1. *Linearity property*

$$F_c\left[c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)\right]$$
$$= c_1 F_c\left[f_1(x)\right] + c_2 F_c\left[f_2(x)\right] + \cdots + c_n F_c\left[f_n(x)\right]$$

2. *Change of scale property*

If $\quad F_c[f(x)] = F_c(u)$ then $F_c[f(ax)] = \frac{1}{a}F_c\left(\frac{u}{a}\right)$

These two properties continue to hold good in the case of Fourier sine transform also.

3. *Modulation properties*

If $\quad F_s[f(x)] = F_s(u)$ *and* $F_c[f(x)] = F_c(u)$ *then*

(i) $\quad F_s[f(x)\cos ax] = \frac{1}{2}\left[F_s(u+a) + F_s(u-a)\right]$

(ii) $\quad F_s[f(x)\sin ax] = \frac{1}{2}\left[F_c(u-a) - F_c(u+a)\right]$

(iii) $\quad F_c[f(x)\cos ax] = \frac{1}{2}\left[F_c(u+a) + F_c(u-a)\right]$

(iv) $\quad F_c[f(x)\sin ax] = \frac{1}{2}\left[F_s(u+a) - F_s(u-a)\right]$

Proof :

(i) $$F_s[f(x)\cos ax] = \int_0^\infty f(x)\cos ax \cdot \sin ux\, dx$$

$$= \int_0^\infty f(x)\cdot\frac{1}{2}\left[\sin(u+a)x+\sin(u-a)x\right]dx$$

$$= \frac{1}{2}\left[\int_0^\infty f(x)\sin(u+a)x\,dx + \int_0^\infty f(x)\sin(u-a)x\,dx\right]$$

$$\mathbf{F_s[f(x)\cos ax] = \frac{1}{2}\left[F_s(u+a)+F_s(u-a)\right]}$$

(ii) $$F_s[f(x)\sin ax] = \int_0^\infty f(x)\sin ax\cdot\sin ux\,dx$$

$$= \int_0^\infty f(x)\cdot\frac{1}{2}[\cos(u-a)x-\cos(u+a)x]\,dx$$

$$= \frac{1}{2}\left[\int_0^\infty f(x)\cos(u-a)x\,dx - \int_0^\infty f(x)\cos(u+a)x\,dx\right]$$

$$\mathbf{F_s[f(x)\sin ax] = \frac{1}{2}\left[F_c(u-a)-F_c(u+a)\right]}$$

(iii) $$F_c[f(x)\cos ax] = \int_0^\infty f(x)\cos ax\cdot\cos ux\,dx$$

$$= \int_0^\infty f(x)\cdot\frac{1}{2}[\cos(u+a)x+\cos(u-a)x]\,dx$$

$$= \frac{1}{2}\left[\int_0^\infty f(x)\cos(u+a)x\,dx + \int_0^\infty f(x)\cos(u-a)x\,dx\right]$$

$$\mathbf{F_c[f(x)\cos ax] = \frac{1}{2}\left[F_c(u+a)+F_c(u-a)\right]}$$

(iv) $$F_c[f(x)\sin ax] = \int_0^{\infty} f(x)\sin ax \cdot \cos ux\, dx$$

$$= \int_0^{\infty} f(x)\cdot\frac{1}{2}\left[\sin(a+u)x + \sin(a-u)x\right]dx$$

$$= \frac{1}{2}\left[\int_0^{\infty} f(x)\sin(u+a)x\,dx - \int_0^{\infty} f(x)\sin(u-a)x\,dx\right]$$

$$F_c[f(x)\sin ax] = \frac{1}{2}\left[F_s(u+a) - F_s(u-a)\right]$$

Definitions at a glance - Infinite Fourier transforms

Type	Transform	Inverse transform
Fourier transform	$\int_{-\infty}^{\infty} f(x)e^{iux}dx = F(u)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{-iux}du = f(x)$
Fourier cosine transform	$\int_0^{\infty} f(x)\cos ux\,dx = F_c(u)$	$\frac{2}{\pi}\int_0^{\infty} F_c(u)\cos ux\,du = f(x)$
Fourier sine transform	$\int_0^{\infty} f(x)\sin ux\,dx = F_s(u)$	$\frac{2}{\pi}\int_0^{\infty} F_s(u)\sin ux\,du = f(x)$

Note : **Definitions in the alternative / equivalent form.**

Type	Transform	Inverse transform
Fourier transform	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{iux}dx = F(u)$	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(u)e^{-iux}du = f(x)$
Fourier cosine transform	$\sqrt{2/\pi}\int_0^{\infty} f(x)\cos ux\,dx = F_c(u)$	$\sqrt{2/\pi}\int_0^{\infty} F_c(u)\cos ux\,du = f(x)$
Fourier sine transform	$\sqrt{2/\pi}\int_0^{\infty} f(x)\sin ux\,dx = F_s(u)$	$\sqrt{2/\pi}\int_0^{\infty} F_s(u)\sin ux\,du = f(x)$

WORKED PROBLEMS

1. *Find the complex Fourier transform of the function*

$$f(x) = \begin{cases} 1 \text{ for } |x| \le a \\ 0 \text{ for } |x| > a \end{cases} \quad \textit{Hence evaluate} \int_0^\infty \frac{\sin x}{x}\, dx$$

>> Complex Fourier transform of $f(x)$ is given by

$$F(u) = \int_{x=-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

$$= \int_{x=-a}^{a} 1 \cdot e^{iux}\, dx, \qquad \text{since } f(x) = \begin{cases} 1 \text{ for } -a \le x \le a \\ 0 \text{ otherwise} \end{cases}$$

$$F(u) = \left[\frac{e^{iux}}{iu} \right]_{x=-a}^{a} = \frac{1}{iu} \left\{ e^{iua} - e^{-iua} \right\}$$

$$F(u) = \frac{1}{iu} \left\{ (\cos au + i \sin au) - (\cos au - i \sin au) \right\}$$

$$= \frac{1}{iu} (2i \sin au) = \frac{2 \sin au}{u}$$

Thus $\mathbf{F(u) = \frac{2 \sin au}{u}}$.

Let us evaluate $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx$

We have obtained $F(u) = \dfrac{2 \sin au}{u}$

Inverse Fourier transform is $\displaystyle\frac{1}{2\pi} \int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du = f(x)$

i.e., $$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin au}{u}\, e^{-iux}\, du = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin au}{u}\, e^{-iux}\, du$$

Now let us put $x = 0$.

Since $x = 0$ is a point of continuity of $f(x)$, the value of $f(x)$ at $x = 0$ being $f(0) = 1$ because $f(x) = 1$ for $|x| \le a$

Hence $\displaystyle\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin au}{u}\, du = 1$ since $e^0 = 1$

i.e., $\dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\sin au}{u}\,du = 1$, since $\dfrac{\sin au}{u}$ is an even function of u

$$\therefore \quad \int_0^\infty \frac{\sin au}{u}\,du = \frac{\pi}{2}$$

Putting $a = 1$, $\displaystyle\int_0^\infty \frac{\sin u}{u}\,du = \frac{\pi}{2}$

Thus by changing u to x, we have $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

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2. *Find the complex Fourier transform of the function*

$$f(x) = \begin{cases} x, & |x| \le \alpha \\ 0, & |x| > \alpha \end{cases} \quad \textit{where } \alpha \textit{ is a positive constant.}$$

\>> $F(u) = \displaystyle\int_{x=-\infty}^{\infty} f(x)\, e^{iux}\,dx$ and by data $f(x) = x$ for $|x| \le \alpha$

$$= \int_{-\alpha}^{\alpha} x \cdot e^{iux}\,dx$$

$$= \left[x \cdot \frac{e^{iux}}{iu} - 1 \cdot \frac{e^{iux}}{i^2 u^2} \right]_{-\alpha}^{\alpha}, \text{ by Bernoulli's rule.}$$

$$= \frac{1}{iu}\left[x e^{iux} \right]_{-\alpha}^{\alpha} - \frac{1}{i^2 u^2}\left[e^{iux} \right]_{-\alpha}^{\alpha}.$$

$$= \frac{-i}{u}\left\{ \alpha\, e^{iu\alpha} - (-\alpha)\, e^{-iu\alpha} \right\} + \frac{1}{u^2}\left\{ e^{iu\alpha} - e^{-iu\alpha} \right\}$$

Also $e^{iu\alpha} = \cos u\alpha + i \sin u\alpha$, $e^{-iu\alpha} = \cos u\alpha - i\sin u\alpha$.

$$\therefore \quad e^{iu\alpha} + e^{-iu\alpha} = 2\cos u\alpha, \quad e^{iu\alpha} - e^{-iu\alpha} = 2i\sin u\alpha$$

Hence $F(u) = \dfrac{-i\,2\alpha\cos\alpha u}{u} + \dfrac{i\,2\sin\alpha u}{u^2}$

Thus $\quad \boldsymbol{F(u) = 2i\left(\dfrac{\sin \alpha u}{u^2} - \dfrac{\alpha\cos\alpha u}{u} \right)}$

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3. *If* $f(x) = \begin{cases} 1-x^2, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$

find the Fourier transform of $f(x)$ *and hence find the value of*

(i) $\displaystyle\int_0^{\infty} \frac{x\cos x - \sin x}{x^3}\, dx.$ **(ii)** $\displaystyle\int_0^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx$

>> $$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx = \int_{-1}^{1} (1-x^2)\, e^{iux}\, dx,$$

$\because$ $f(x) = 0$ for $|x| \geq 1$ and $1-x^2$ for $|x| < 1$

$$\therefore \quad F(u) = \left[(1-x^2)\frac{e^{iux}}{iu} - (-2x)\frac{e^{iux}}{i^2u^2} + (-2)\frac{e^{iux}}{i^3u^3} \right]_{x=-1}^{1}$$ by Bernoulli's rule.

$$= \frac{-i}{u}\left[(1-x^2)\,e^{iux}\right]_{x=-1}^{1} - \frac{2}{u^2}\left[x\,e^{iux}\right]_{x=-1}^{1} - \frac{2i}{u^3}\left[e^{iux}\right]_{x=-1}^{1}$$

$\left(i^2 = -1,\ \frac{1}{i} = -i,\ \frac{1}{i^3} = i\right)$

$$F(u) = \frac{-i}{u}(0-0) - \frac{2}{u^2}\left\{1 \cdot e^{iu} - (-1)\,e^{-iu}\right\} - \frac{2i}{u^3}\left(e^{iu} - e^{-iu}\right)$$

$$= -\frac{2}{u^2}\left(e^{iu} + e^{-iu}\right) - \frac{2i}{u^3}\left(e^{iu} - e^{-iu}\right)$$

But $e^{iu} = \cos u + i\sin u,\quad e^{-iu} = \cos u - i\sin u$

$\therefore$ $e^{iu} + e^{-iu} = 2\cos u,\quad e^{iu} - e^{-iu} = 2i\sin u$

Hence $F(u) = \dfrac{-4\cos u}{u^2} + \dfrac{4\sin u}{u^3}$

Thus $\boldsymbol{F(u) = 4\left(\dfrac{\sin u - u\cos u}{u^3}\right)}$

Let us evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x\cos x - \sin x}{x^3}\, dx$

By inverse Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du \qquad \ldots(1)$$

If $x = 0$: $f(x) = 1 - 0^2 = 1$ at $x = 0$. By putting $x = 0$ in the integral and using the expression of $F(u)$ we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} 4\left(\frac{\sin u - u\cos u}{u^3}\right) e^0\, du = f(0) = 1.$$

$$\int_{-\infty}^{\infty} \frac{\sin u - u\cos u}{u^3}\, du = \frac{2\pi}{4} = \frac{\pi}{2}$$

If u is changed to $-u$, the expression $\dfrac{\sin u - u\cos u}{u^3}$ becomes

$\dfrac{\sin(-u) - (-u)\cos(-u)}{(-u)^3} = \dfrac{\sin u - u\cos u}{u^3}$ itself. Therefore the function is even and hence the integral from $-\infty$ to ∞ is twice the integral from 0 to ∞.

$$\therefore \quad 2\int_0^{\infty} \frac{\sin u - u\cos u}{u^3}\, du = \frac{\pi}{2} \quad \text{or} \quad \int_0^{\infty} \frac{u\cos u - \sin u}{u^3}\, du = -\frac{\pi}{4}$$

Changing u to x we get $\displaystyle \int_0^{\infty} \frac{x\cos x - \sin x}{x^3}\, dx = -\frac{\pi}{4}$

Next, let us evaluate $\displaystyle \int_0^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx$

Putting $x = 1/2$ in (1) we have,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} 4\left(\frac{\sin u - u\cos u}{u^3}\right) e^{\frac{-iu}{2}}\, du = f\left(\frac{1}{2}\right) = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$$

ie., $$\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin u - u\cos u}{u^3}\left(\cos\frac{u}{2} - i\sin\frac{u}{2}\right) du = \frac{3}{4}$$

Equating the real parts on both sides we get,

$$\int_{-\infty}^{\infty} \frac{\sin u - u\cos u}{u^3} \cos\frac{u}{2}\, du = \frac{3\pi}{8}$$

i.e., $$2\int_0^{\infty} \frac{\sin u - u\cos u}{u^3} \cos\frac{u}{2}\, du = \frac{3\pi}{8},$$ since the integrand is even.

Dividing by 2, changing the sign and writing x in place of u we get,

$$\int_0^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\, dx = \frac{-3\pi}{16}$$

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4. *Find the Fourier transform of* $f(x) = e^{-|x|}$

>> Fourier transform of $f(x)$ is given by $F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$

Here $f(x) = e^{-|x|} = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ e^{x} & \text{for } x < 0 \end{cases}$

$$\therefore \quad F(u) = \int_{-\infty}^{0} e^{x} e^{iux}\, dx + \int_{0}^{\infty} e^{-x} e^{iux}\, dx$$

$$F(u) = \int_{-\infty}^{0} e^{(1+iu)x}\, dx + \int_{0}^{\infty} e^{-(1-iu)x}\, dx$$

$$= \left[\frac{e^{(1+iu)x}}{1+iu} \right]_{x=-\infty}^{0} + \left[\frac{e^{-(1-iu)x}}{-(1-iu)} \right]_{x=0}^{\infty}$$

$$= \left[\frac{1}{1+iu} - 0 \right] + \left[0 - \frac{1}{-(1-iu)} \right]$$

$$= \frac{1}{1+iu} + \frac{1}{1-iu} = \frac{2}{1-i^2 u^2} = \frac{2}{1+u^2}$$

Thus $\boldsymbol{F(u) = \dfrac{2}{1+u^2}}$

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5. *Find the Fourier transform of*

$$f(x) = \begin{cases} 1 - |x| & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases} \quad \text{and hence deduce that} \int_{0}^{\infty} \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}$$

>> $$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx = \int_{-1}^{1} 1 - |x| \; e^{iux}\, dx$$

$$F[f(x)] = \int_{-1}^{0} [1-(-x)]\, e^{iux}\, dx + \int_{0}^{1} [1-(+x)]\, e^{iux}\, dx$$

$$= \int_{-1}^{0} (1+x)\, e^{iux}\, dx + \int_{0}^{1} (1-x)\, e^{iux}\, dx$$

$$F[f(x)] = \left[(1+x)\frac{e^{iux}}{iu} - (1)\frac{e^{iux}}{(iu)^2} \right]_{-1}^{0} + \left[(1-x)\frac{e^{iux}}{iu} - (-1)\frac{e^{iux}}{(iu)^2} \right]_{0}^{1}$$

$$= \frac{1}{iu}(1-0) + \frac{1}{u^2}\left(1-e^{-iu}\right) + \frac{1}{iu}(0-1) - \frac{1}{u^2}\left(e^{iu}-1\right)$$

$$= \frac{2}{u^2} - \frac{1}{u^2}\left(e^{iu}+e^{-iu}\right) = \frac{2}{u^2} - \frac{2\cos u}{u^2} = \frac{2(1-\cos u)}{u^2}$$

Thus $\quad \boldsymbol{F[f(x)] = \frac{4\sin^2(u/2)}{u^2} = F(u)}$

Now $f(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du$, being the inverse Fourier transform.

i.e., $\quad f(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{4\sin^2(u/2)}{u^2}\, e^{-iux}\, du$

i.e., $\quad f(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2(u/2)}{(u/2)^2}\, e^{-iux}\, du$

Putting $x = 0$ we have $f(0) = 1$ by the definition of $f(x)$.

$\therefore \quad 1 = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2(u/2)}{(u/2)^2}\, du$

Put $\quad u/2 = t \quad \therefore \quad du = 2\,dt$ and t also varies from $-\infty$ to ∞

Hence, $1 = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2}\cdot 2\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2}\, dt$

or $\quad 1 = \dfrac{1}{\pi}\cdot 2\displaystyle\int_{0}^{\infty} \frac{\sin^2 t}{t^2}\, dt$, since the integrand is even.

Thus $\quad \boldsymbol{\displaystyle\int_{0}^{\infty} \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}}$

6. *Find the Fourier transform of*

$$f(x) = \begin{cases} x^2, & |x| < a \\ 0, & |x| > a \end{cases} \quad \text{where } a \text{ is a positive constant.}$$

>> Fourier transform of $f(x)$ is given by $F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$

Since $f(x) = x^2$ for $|x| < a$ by data, we have

$$F(u) = \int_{-a}^{a} x^2 e^{iux}\, dx.$$

$$= \left[(x^2) \frac{e^{iux}}{iu} - (2x) \frac{e^{iux}}{i^2 u^2} + (2) \frac{e^{iux}}{i^3 u^3} \right]_{-a}^{a}$$

$$= \frac{1}{iu} \left(a^2 e^{iua} - a^2 e^{-iua} \right) + \frac{2}{u^2} \left(a\, e^{iua} + a\, e^{-iua} \right) - \frac{2}{iu^3} \left(e^{iua} - e^{-iua} \right)$$

$$= \frac{a^2}{iu} (2i \sin au) + \frac{2a}{u^2} (2 \cos au) - \frac{2}{iu^3} (2i \sin au)$$

$$= \frac{2a^2 \sin au}{u} + \frac{4a \cos au}{u^2} - \frac{4 \sin au}{u^3}$$

Thus $\mathbf{F(u) = \frac{1}{u^3} \left[2(a^2 u^2 - 2) \sin au + 4\, au \cos au \right]}$

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7. *Find the Fourier transform of* $f(x) = x\, e^{-|x|}$

>> We have $F(u) = F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$

$$f(x) = x\, e^{-|x|} = \begin{cases} x\, e^{x} & \text{for } x < 0 \\ x\, e^{-x} & \text{for } x > 0 \end{cases}$$

$$F(u) = \int_{-\infty}^{0} f(x)\, e^{iux}\, dx + \int_{0}^{\infty} f(x)\, e^{iux}\, dx$$

$$= \int_{-\infty}^{0} x\, e^{x} e^{iux}\, dx + \int_{0}^{\infty} x\, e^{-x} e^{iux}\, dx$$

$$F(u) = \int_{-\infty}^{0} x\, e^{(1+iu)x}\, dx + \int_{0}^{\infty} x\, e^{-(1-iu)x}\, dx$$

Applying Bernoulli's rule to each of the integrals,

$$F(u) = \left[(x)\frac{e^{(1+iu)x}}{(1+iu)} - (1)\frac{e^{(1+iu)x}}{(1+iu)^2} \right]_{-\infty}^{0} + \left[(x)\frac{e^{-(1-iu)x}}{-(1-iu)} - (1)\frac{e^{-(1-iu)x}}{(1-iu)^2} \right]_{0}^{\infty}$$

The first and third terms vanish.

$$F(u) = \frac{-1}{(1+iu)^2}(1-0) - \frac{1}{(1-iu)^2}(0-1)$$

$$= \frac{1}{(1-iu)^2} - \frac{1}{(1+iu)^2} = \frac{(1+iu)^2-(1-iu)^2}{(1+u^2)^2}$$

Thus $\mathbf{F(u) = \dfrac{4iu}{(1+u^2)^2}}$

---

8. *Find the Fourier transform of*

$$f(x) = \begin{cases} 1+(x/a), & -a < x < 0 \\ 1-(x/a), & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

>> $F(u) = F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$

ie., $F(u) = \int_{-a}^{a} f(x)\, e^{iux}\, dx = \int_{-a}^{0} f(x)\, e^{iux}\, dx + \int_{0}^{a} f(x)\, e^{iux}\, dx$

$$F(u) = \int_{-a}^{0} \left(1+\frac{x}{a}\right) e^{iux}\, dx + \int_{0}^{a} \left(1-\frac{x}{a}\right) e^{iux}\, dx$$

$$= \left[\left(1+\frac{x}{a}\right)\frac{e^{iux}}{iu} - \left(\frac{1}{a}\right)\frac{e^{iux}}{i^2u^2} \right]_{-a}^{0} + \left[\left(1-\frac{x}{a}\right)\frac{e^{iux}}{iu} - \left(\frac{-1}{a}\right)\frac{e^{iux}}{i^2u^2} \right]_{0}^{a}$$

$$= \frac{1}{iu}(1-0) + \frac{1}{au^2}\left(1-e^{-iua}\right) + \frac{1}{iu}(0-1) - \frac{1}{au^2}\left(e^{iua}-1\right)$$

$$= \frac{1}{au^2}\left[2 - \left(e^{iau} + e^{-iau}\right)\right] = \frac{2(1-\cos au)}{au^2}$$

Thus $\quad \boldsymbol{F(u) = \dfrac{4\sin^2(au/2)}{au^2}}$

---

9. *Find the complex Fourier transform of $e^{-a^2x^2}$, $a > 0$. Hence deduce that $e^{-x^2/2}$ is self reciprocal in respect of the complex Fourier transform.*

>> $\quad F(u) = F[f(x)] = \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$

i.e., $\quad F(u) = \displaystyle\int_{-\infty}^{\infty} e^{-a^2x^2} \cdot e^{iux}\, dx = \int_{-\infty}^{\infty} e^{-a^2\left(x^2 - \frac{iux}{a^2}\right)}\, dx$

$$= \int_{-\infty}^{\infty} e^{-a^2\left(x^2 - 2x\cdot\frac{iu}{2a^2} + \frac{i^2u^2}{4a^4} - \frac{i^2u^2}{4a^4}\right)}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2\left(x - \frac{iu}{2a^2}\right)^2} \cdot e^{-u^2/4a^2}\, dx$$

Put $\quad a\left(x - \dfrac{iu}{2a^2}\right) = t \quad \therefore\ dx = \dfrac{dt}{a}$ and t also varies from $-\infty$ to ∞

Now $\quad F(u) = e^{-u^2/4a^2} \displaystyle\int_{-\infty}^{\infty} e^{-t^2} \cdot \frac{dt}{a}$ $\quad$ and we know that $\displaystyle\int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}$

Thus $\quad \boldsymbol{F(u) = \dfrac{\sqrt{\pi}}{a}\, e^{-u^2/4a^2}}$

Now taking $a^2 = 1/2$ we have

$$F(u) = F\left[e^{-x^2/2}\right] = \frac{\sqrt{\pi}}{(1/\sqrt{2})}\, e^{-u^2/2} = \sqrt{2\pi}\, e^{-u^2/2}$$

It can be seen that the Fourier transform of $e^{-x^2/2}$ is a constant times $e^{-u^2/2}$. The function $e^{-x^2/2}$ and $e^{-u^2/2}$ are same but for the change in the variable.

Hence we conclude that $e^{-x^2/2}$ is self reciprocal under complex Fourier transform.

---

10. *Find the inverse Fourier transform of* e^{-u^2}

>> We have the inverse Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-u^2}\cdot e^{-iux}\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(u^2+iux)}\, du$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(u^2+2\cdot u\cdot\frac{ix}{2}+\frac{i^2x^2}{4}-\frac{i^2x^2}{4}\right)}\, du$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(u+\frac{ix}{2}\right)^2}\cdot e^{-x^2/4}\, du$$

Put $u+\dfrac{ix}{2} = t \quad \therefore \quad du = dt$ and t varies from $-\infty$ to ∞

Hence $f(x) = \dfrac{e^{-x^2/4}}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{-t^2}\, dt = \dfrac{e^{-x^2/4}}{2\pi}\sqrt{\pi} = \dfrac{e^{-x^2/4}}{2\sqrt{\pi}}$

Thus the required inverse Fourier transform is $\dfrac{e^{-x^2/4}}{2\sqrt{\pi}}$

---

11. *Find the Fourier sine and cosine transforms of*

$$f(x) = \begin{cases} x, & 0 < x < 2 \\ 0, & else\ where \end{cases}$$

>> The Fourier sine and cosine transforms of $f(x)$ are given by

$$F_s(u) = \int_0^{\infty} f(x)\sin ux\, dx \quad\text{and}\quad F_c(u) = \int_0^{\infty} f(x)\cos ux\, dx$$

$\therefore \quad F_s(u) = \displaystyle\int_0^2 x\sin ux\, dx$

$$= \left[x\cdot\frac{-\cos ux}{u} - 1\cdot\frac{-\sin ux}{u^2}\right]_0^2 \quad \text{by Bernoulli's rule.}$$

$$F_s(u) = -\frac{1}{u}\left[x \cos ux\right]_0^2 + \frac{1}{u^2}\left[\sin ux\right]_0^2$$

$$= -\frac{1}{u}(2\cos 2u - 0) + \frac{1}{u^2}(\sin 2u - \sin 0)$$

Thus $\quad F_s(u) = \dfrac{\sin 2u - 2u\cos 2u}{u^2}$

Also $\quad F_c(u) = \displaystyle\int_0^2 x \cos ux\, dx$

$$= \left[x\frac{\sin ux}{u} - 1 \cdot \frac{-\cos ux}{u^2}\right]_0^2 \text{ by Bernoulli's rule.}$$

$$= \frac{1}{u}\left[x \sin ux\right]_0^2 + \frac{1}{u^2}\left[\cos ux\right]_0^2$$

$$= \frac{1}{u}(2\sin 2u - 0) + \frac{1}{u^2}(\cos 2u - \cos 0)$$

$$= \frac{2\sin 2u}{u} + \frac{\cos 2u - 1}{u^2},$$

Thus $\quad F_c(u) = \dfrac{2u\sin 2u + \cos 2u - 1}{u^2}$

---

12. *Find the Fourier sine and cosine transforms of* $f(x) = e^{-\alpha x},\ \alpha > 0$

\>> Fourier sine and cosine transforms are given by

$$F_s(u) = \int_0^\infty f(x)\sin ux\, dx \quad \text{and} \quad F_c(u) = \int_0^\infty f(x)\cos ux\, dx$$

$$\therefore \quad F_s(u) = \int_0^\infty e^{-\alpha x}\sin ux\, dx$$

$$= \left[\frac{e^{-\alpha x}}{(-\alpha)^2 + u^2}(-\alpha \sin ux - u\cos ux)\right]_0^\infty$$

by using the standard formula,

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2}$$

But $\quad e^{-\alpha x} \to 0$ as $x \to \infty$, $e^0 = 1$, $\cos 0 = 1$, and $\sin 0 = 0$

Thus $\quad F_s(u) = \dfrac{u}{\alpha^2 + u^2}$

Also $\quad F_c(u) = \int_0^\infty f(x) \cos ux\, dx = \int_0^\infty e^{-\alpha x} \cos ux\, dx$

$$= \left[\frac{e^{-\alpha x}}{(-\alpha)^2 + u^2} (-\alpha \cos ux + u \sin ux) \right]_{x=0}^{\infty} = \frac{\alpha}{\alpha^2 + u^2}$$

by using $\int e^{ax} \cos bx\, dx = \dfrac{e^{ax} (a \cos bx + b \sin bx)}{a^2 + b^2}$

Thus $\quad F_c(u) = \dfrac{\alpha}{\alpha^2 + u^2}$

---

13. *Obtain the Fourier cosine transform of the function*

$$f(x) = \begin{cases} 4x, & 0 < x < 1 \\ 4 - x, & 1 < x < 4 \\ 0, & x > 4 \end{cases}$$

\>> Fourier cosine transform is given by

$$F_c(u) = \int_0^\infty f(x) \cos ux\, dx$$

$$= \int_0^1 f(x) \cos ux\, dx + \int_1^4 f(x) \cos ux\, dx + \int_4^\infty f(x) \cos ux\, dx$$

$$\therefore \quad F_c(u) = \int_0^1 4x \cos ux\, dx + \int_1^4 (4 - x) \cos ux\, dx + \int_4^\infty 0 \cdot \cos ux\, dx$$

Applying Bernoulli's rule to the integrals we have,

$$F_c(u) = \left[4x \cdot \frac{\sin ux}{u} - 4\,\frac{-\cos ux}{u^2} \right]_0^1 + \left[(4-x)\,\frac{\sin ux}{u} - (-1)\,\frac{-\cos ux}{u^2} \right]_1^4 + 0$$

$$= \frac{4}{u}\,[\,x \sin ux\,]_0^1 + \frac{4}{u^2}\,[\,\cos ux\,]_0^1 + \frac{1}{u}\left[(4-x)\sin ux \right]_1^4 - \frac{1}{u^2}\,[\,\cos ux\,]_1^4$$

$$= \frac{4}{u}\left(\sin u - 0\right) + \frac{4}{u^2}\left(\cos u - 1\right) + \frac{1}{u}\left(0 - 3\sin u\right) - \frac{1}{u^2}\left(\cos 4u - \cos u\right)$$

$$= \frac{4}{u}\sin u + \frac{4}{u^2}\cos u - \frac{4}{u^2} - \frac{3}{u}\sin u - \frac{1}{u^2}\cos 4u + \frac{1}{u^2}\cos u$$

Thus $\quad \mathbf{F_c(u) = \frac{1}{u}\sin u + \frac{5\cos u - 4}{u^2} - \frac{1}{u^2}\cos 4u}$

---

14. *Find the infinite Fourier cosine transform of* e^{-x^2}

>> Fourier cosine transform is given by

$$F_c(u) = \int_0^\infty f(x)\,\cos ux\,dx$$

$$F_c(u) = \int_0^\infty e^{-x^2} \cos ux\,dx \qquad \ldots (1)$$

Note : *The integral is to be evaluated by using Leibnitz rule for differentiation under the integral sign.*

Differentiating w.r.t. u using Leibnitz rule, we have

$$\frac{dF_c}{du} = \int_0^\infty \frac{\partial}{\partial u}\left(e^{-x^2}\cos ux\right) dx$$

$$= \int_0^\infty e^{-x^2}\,(-\sin ux \cdot x)\,dx = \frac{1}{2}\int_0^\infty \sin ux\,\left\{ e^{-x^2}\,(-2x) \right\} dx$$

or $\quad 2\,\dfrac{dF_c}{du} = \displaystyle\int_0^\infty \sin ux \left\{ e^{-x^2}\left(-2x\right) \right\} dx$

Integrating R.H.S by parts we have,

$$2\,\frac{dF_c}{du} = \left[\sin ux\,(e^{-x^2}) \right]_0^\infty - \int_0^\infty e^{-x^2}\,(\cos ux \cdot u)\,dx$$

But $e^{-x^2} \to 0$ as $x \to \infty$ and $\sin 0 = 0$

$$\therefore \quad 2\frac{dF_c}{du} = (0-0) - u\int_0^\infty e^{-x^2}\cos ux\,dx \quad \text{or} \quad 2\frac{dF_c}{du} = -u F_c$$

i.e., $\quad 2\dfrac{dF_c}{F_c} = -u\,du$

or $\quad \dfrac{dF_c}{F_c} = -\dfrac{u}{2}\,du$ and integration yields

$$\log F_c = \frac{-u^2}{4} + \log k, \text{ where } \log k \text{ is a constant.}$$

ie., $\quad \log\left(\dfrac{F_c}{k}\right) = \dfrac{-u^2}{4}$ or $\dfrac{F_c}{k} = e^{-u^2/4}$

Hence we have, $F_c(u) = k\,e^{-u^2/4}$ . . . (2)

To find k let us put $u = 0$ in (1) and (2).

(1) gives $F_c(0) = \displaystyle\int_0^\infty e^{-x^2}\cos 0\,dx = \int_0^\infty e^{-x^2}\,dx$

But $\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$ and hence $F_c(0) = \dfrac{\sqrt{\pi}}{2}$

Now putting $u = 0$ in (2), $F_c(0) = k\,e^0 = k$. From these we get $k = \sqrt{\pi}/2$.

Thus by substituitng the value of k in (2) we have

$$\mathbf{F_c(u) = (\sqrt{\pi}/2)\,e^{-u^2/4}}$$

---

15. *Find the Fourier sine transform of* $f(x) = e^{-|x|}$ *and hence evaluate* $\displaystyle\int_0^\infty \frac{x\sin mx}{1+x^2}\,dx,\ m > 0$

>> Fourier sine transform is given by

$$F_s(u) = \int_0^\infty f(x)\,\sin ux\,dx$$

$$F_s(u) = \int_0^\infty e^{-|x|} \sin ux\, dx = \int_0^\infty e^{-x} \sin ux\, dx, \quad \text{since } |x| = x, \; x > 0$$

$$F_s(u) = \left[\frac{e^{-x}(-1 \sin ux - u \cos ux)}{(-1)^2 + u^2} \right]_0^\infty$$

But $\quad e^{-x} \to 0$ as $x \to \infty$, $e^0 = 1$, $\cos 0 = 1$, $\sin 0 = 0$

Thus $\quad \mathbf{F_s(u) = \dfrac{u}{1+u^2}}$

By inverse Fourier sine transform we have

$$\frac{2}{\pi} \int_0^\infty F_s(u) \sin ux\, du = f(x)$$

i.e.,

$$\int_0^\infty \frac{u}{1+u^2} \sin ux\, du = \frac{\pi}{2} f(x)$$

Putting $x = m$ where $m > 0$ we have $f(x) = e^{-|m|} = e^{-m}$

$$\int_0^\infty \frac{u \sin mu}{1+u^2}\, du = \frac{\pi}{2} e^{-m}$$

Thus by changing the variable u to x, $\quad \displaystyle\int_0^\infty \frac{x \sin mx}{1+x^2}\, dx = \frac{\pi}{2} e^{-m}$

---

16. *Find $f(x)$ if its Fourier sine transform is $p^n e^{-ap}$, $a > 0$*

>> Here $F_s(p) = p^n e^{-ap}$

By inverse Fourier sine transform

$$f(x) = \frac{2}{\pi} \int_{p=0}^\infty p^n e^{-ap} \sin px\, dp \qquad \ldots (1)$$

To evaluate this integral consider $\displaystyle\int_0^\infty e^{-ap} \sin px\, dp$

We know that $\int e^{ax} \sin bx \, dx = \dfrac{e^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2}$

$$\therefore \qquad \int_0^\infty e^{-ap} \sin px \, dp = \left[\frac{e^{-ap} (-a \sin xp - x \cos xp)}{(-a)^2 + x^2} \right]_0^\infty$$

But $\quad e^{-ap} \to 0$ as $p \to \infty$, $e^0 = 1$, $\cos 0 = 1$, $\sin 0 = 0$

Hence $\displaystyle\int_0^\infty e^{-ap} \sin px \, dx = \frac{x}{a^2 + x^2}$

Differentiating n times w.r.t a using Leibnitz rule,

$$\frac{d^n}{da^n} \int_0^\infty e^{-ap} \sin px \, dp = \frac{d^n}{da^n} \left(\frac{x}{a^2 + x^2} \right)$$

i.e., $$\int_0^\infty e^{-ap} (-p)^n \sin px \, dp = \frac{d^n}{da^n} \left(\frac{x}{a^2 + x^2} \right) \qquad \ldots (2)$$

We can write

$$\frac{x}{a^2 + x^2} = \frac{x}{(a + ix) (a - ix)} = \frac{1}{2i} \left(\frac{-1}{a + ix} + \frac{1}{a - ix} \right),$$

by resolving into partial fractions.

$$\therefore \qquad \frac{d^n}{da^n} \left(\frac{x}{a^2 + x^2} \right) = \frac{-1}{2i} \frac{d^n}{da^n} \left(\frac{1}{a + ix} \right) + \frac{1}{2i} \frac{d^n}{da^n} \left(\frac{1}{a - ix} \right)$$

$$= \frac{-1}{2i} \frac{(-1)^n \, n!}{(a + ix)^{n+1}} + \frac{1}{2i} \frac{(-1)^n \, n!}{(a - ix)^{n+1}},$$

by using the formula $\dfrac{d^n}{dx^n} \left(\dfrac{1}{ax + b} \right) = \dfrac{(-1)^n \, n! \, a^n}{(ax + b)^{n+1}}$

i.e., $$\frac{d^n}{da^n} \left(\frac{x}{a^2 + x^2} \right) = \frac{(-1)^n \, n!}{2i} \left\{ \frac{-1}{(a + ix)^{n+1}} + \frac{1}{(a - ix)^{n+1}} \right\}$$

i.e., $$\frac{d^n}{da^n} \left(\frac{x}{a^2 + x^2} \right) = \frac{(-1)^n \, n!}{2i} \left[- (a + ix)^{-(n+1)} + (a - ix)^{-(n+1)} \right] \qquad \ldots (3)$$

Putting $a = r\cos\theta,\ x = r\sin\theta$, we have

$$(a + ix)^{-(n+1)} = [r(\cos\theta + i\sin\theta)]^{-(n+1)}$$
$$= r^{-(n+1)}[\cos(n+1)\theta - i\sin(n+1)\theta] \quad \ldots(4)$$

$$(a - ix)^{-(n+1)} = [r(\cos\theta - i\sin\theta)]^{-(n+1)}$$
$$= r^{-(n+1)}[\cos(n+1)\theta + i\sin(n+1)\theta] \quad \ldots(5)$$

(5) − (4) will give us $r^{-(n+1)} \cdot 2i\ \sin(n+1)\ \theta$ and by substituting in (3) we get

$$\frac{d^n}{da^n}\left(\frac{x}{a^2 + x^2}\right) = \frac{(-1)^n n!}{r^{n+1}} \sin(n+1)\theta$$

Substituting in (2)

$$\int_0^\infty e^{-ap}(-1)^n p^n \sin px\, dp = \frac{(-1)^n n!}{r^{n+1}} \sin(n+1)\theta$$

or
$$\int_0^\infty e^{-ap} p^n \sin px\, dp = \frac{n!}{r^{n+1}} \sin(n+1)\theta$$

Substituting this in (1), we have

$$f(x) = \frac{2}{\pi}\frac{n!}{r^{n+1}} \sin(n+1)\theta \quad \text{where } r\cos\theta = a \text{ and } r\sin\theta = x$$

Further we have $r = \sqrt{a^2 + x^2}$ and $\tan\theta = x/a$

Thus $$f(x) = \frac{2}{\pi}\frac{n!}{(a^2+x^2)^{\frac{n+1}{2}}} \sin\left[(n+1)\tan^{-1}\left(\frac{x}{a}\right)\right]$$

---

17. *If the Fourier sine transform of $f(x)$ is given by $F_s(u) = (\pi/2)\, e^{-2u}$ find the function $f(x)$*

>> By data, $F_s(u) = (\pi/2)\, e^{-2u}$,

By inverse Fourier sine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(u) \sin ux\, du$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\pi}{2}\, e^{-2u} \sin ux\, du$$

But $\int e^{ax} \sin bx\, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$

$$\therefore \quad f(x) = \left[\frac{e^{-2u}\,(-2 \sin xu - x \cos xu)}{(-2)^2 + x^2} \right]_{u=0}^{\infty}$$

$$= \frac{-1}{4+x^2}\left[e^{-2u}\,(2 \sin xu + x \cos xu) \right]_{u=0}^{\infty}$$

But $e^{-2u} \to 0$ as $u \to \infty$, $e^0 = 1$, $\cos 0 = 1$, $\sin 0 = 0$

Hence $f(x) = \frac{-1}{(4+x^2)} \cdot [0 - x] = \frac{x}{(4+x^2)}$

Thus $\boldsymbol{f(x) = \frac{x}{(4+x^2)}}$

---

18. *Solve the integral equation*

$$\int_0^\infty f(\theta) \cos \alpha\theta \, d\theta = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases} \quad \textit{and hence evaluate} \int_0^\infty \frac{\sin^2 t}{t^2}\, dt$$

>> Here we have to find $f(\theta)$ and we shall consider the inverse Fourier cosine transform with $F(\alpha) = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$

$$f(\theta) = \frac{2}{\pi}\int_0^\infty F(\alpha) \cos \alpha\theta \, d\alpha$$

$$= \frac{2}{\pi}\int_{\alpha=0}^{1} (1-\alpha) \cos \alpha\theta \, d\alpha, \text{ since } F(\alpha) = 0 \text{ for } \alpha > 1$$

$$= \frac{2}{\pi}\left[(1-\alpha)\frac{\sin \alpha\theta}{\theta} - (-1)\left(\frac{-\cos \alpha\theta}{\theta^2} \right) \right]_{\alpha=0}^{1}$$

$$= \frac{2}{\pi\theta}\left[(1-\alpha) \sin \alpha\theta \right]_{\alpha=0}^{1} - \frac{2}{\pi\theta^2}\left[\cos \alpha\theta \right]_{\alpha=0}^{1}$$

$$f(\theta) = \frac{2}{\pi\theta}\left[0-0\right] - \frac{2}{\pi\theta^2}\left[\cos\theta - 1\right]$$

i.e., $$f(\theta) = \frac{2(1-\cos\theta)}{\pi\,\theta^2} = \frac{2\cdot 2\sin^2(\theta/2)}{\pi\,\theta^2} = \frac{4\sin^2(\theta/2)}{\pi\,\theta^2}$$

Hence we now have $\displaystyle\int_0^\infty \frac{4\sin^2(\theta/2)}{\pi\,\theta^2}\cos\alpha\theta\; d\theta = F(\alpha)$

i.e., $$\int_0^\infty \frac{\sin^2(\theta/2)}{\left(\theta/2\right)^2}\cos\alpha\theta\; d\theta = \pi F(\alpha)$$

Putting $\theta/2 = t$, $d\theta = 2\,dt$ and t varies from 0 to ∞.

$$\therefore \quad \int_0^\infty \frac{\sin^2 t}{t^2}\cos(2\alpha t)\cdot 2\,dt = \pi F(\alpha)$$

i.e., $$\int_0^\infty \frac{\sin^2 t}{t^2}\cos(2\alpha t)\; dt = \frac{\pi}{2}F(\alpha)$$

Putting $\alpha = 0$, $F(\alpha) = 1-0 = 1$

Thus $$\int_0^\infty \frac{\sin^2 t}{t^2}\; dt = \frac{\pi}{2}$$

---

19. *Find the Fourier sine transform of* $\dfrac{e^{-ax}}{x}$, $a > 0$

>> We have $F_s(u) = \displaystyle\int_0^\infty f(x)\sin ux\; dx$ and let $f(x) = \dfrac{e^{-ax}}{x}$

i.e., $$F_s(u) = \int_0^\infty \frac{e^{-ax}}{x}\sin ux\; dx \qquad \ldots(1)$$

We cannot evaluate this integral directly and hence we employ the rule of differentiation under the integral sign.

$$\therefore \quad \frac{d}{du}\left[F_s(u)\right] = \int_0^\infty \frac{e^{-ax}}{x}\frac{\partial}{\partial u}(\sin ux)\,dx = \int_0^\infty \frac{e^{-ax}}{x}\,x\cos ux\,dx$$

$$\frac{d}{du}\,[\,F_s(u)\,] = \int_0^\infty e^{-ax}\cos ux\,dx$$

$$= \left[\frac{e^{-ax}}{a^2+u^2}\,(-a\cos ux + u\sin ux)\right]_{x=0}^{\infty} = \frac{1}{a^2+u^2}\,(0+a) = \frac{a}{a^2+u^2}$$

Hence $\frac{d}{du}\left[F_s(u)\right] = \frac{a}{a^2+u^2}$ and by integrating w.r.t u we get,

$$F_s(u) = \tan^{-1}(u/a) + c$$

To evaluate c, let us put $u = 0$ $\quad\therefore\quad F_s(0) = \tan^{-1}(0) + c$

But $\quad F_s(0) = 0$ from (1) and hence $c = 0$.

Thus $\quad \boldsymbol{F_s(u) = \tan^{-1}(u/a)}$

---

20. *Show that* $x\,e^{-x^2/2}$ *is self receprocal under the Fourier sine transform.*

>> By the definition, $F_s\,[f(x)] = \int_0^\infty f(x)\sin ux\,dx = F_s(u)$

Now, $\quad F_s\left[x\,e^{-x^2/2}\right] = \int_0^\infty x\,e^{-x^2/2}\sin ux\,dx$

$$= \int_0^\infty \sin ux\;(x\,e^{-x^2/2})\,dx.$$

Integrating by parts we have,

$$F_s\left[x\,e^{-x^2/2}\right] = \left[\sin ux\;(-e^{-x^2/2})\right]_{x=0}^{\infty} - \int_0^\infty (-e^{-x^2/2})\;u\cos ux\;dx$$

$$= 0 + u\int_0^\infty e^{-x^2/2}\cos ux\;dx$$

$$\therefore \quad F_s(u) = u\int_0^\infty e^{-x^2/2}\cos ux\;dx \qquad \ldots(1)$$

This integral has to be evaluated by Leibnitz rule for differentiation under the integral sign.

Let $\phi(u) = \int_0^\infty e^{-x^2/2} \cos ux \, dx$.

$\therefore \quad \phi'(u) = \int_0^\infty e^{-x^2/2} \frac{\partial}{\partial u}(\cos ux)\, dx$, by Leibnitz rule.

$$\phi'(u) = \int_0^\infty e^{-x^2/2} (-x \sin ux)\, dx$$

$$\phi'(u) = \int_0^\infty \sin ux \,(-x e^{-x^2/2})\, dx$$

Now, integrating by parts we have,

$$\phi'(u) = \left[\sin ux \,(e^{-x^2/2})\right]_{x=0}^{\infty} - \int_0^\infty e^{-x^2/2}(u \cos ux)\, dx$$

$$= 0 - u \int_0^\infty e^{-x^2/2} \cos ux \, dx$$

i.e., $\phi'(u) = -u\,\phi(u)$ or $\frac{\phi'(u)}{\phi(u)} = -u$

$\therefore \quad \int \frac{\phi'(u)}{\phi(u)} du = -\int u \, du + c$

i.e., $\log \phi(u) = (-u^2/2) + c$ or $\phi(u) = e^{(-u^2/2)+c}$

Hence, $\phi(u) = k e^{-u^2/2}$ where $k = e^c$

To evaluate c, let us put $u = 0$

$\therefore \quad \phi(0) = k$ But $\phi(0) = \int_0^\infty e^{-x^2/2} dx$

Put $x/\sqrt{2} = t \quad \therefore \; dx = \sqrt{2}\, dt$

Now $\phi(0) = \int_{t=0}^{\infty} e^{-t^2} \sqrt{2}\, dt.$ But $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$ *(Standard integral)*

Hence $\phi(0) = \sqrt{\pi/2} = k$.

We now have, $\phi(u) = \sqrt{\pi/2}\; e^{-u^2/2}$

Also we have from (1), $F_s(u) = u\,\phi(u) = \sqrt{\pi/2}\; u\,e^{-u^2/2}$

i.e., $$\int_0^\infty x\,e^{-x^2/2}\sin ux\,dx = \sqrt{\pi/2}\,u\,e^{-u^2/2} = \text{const}\cdot(u\,e^{-u^2/2})$$

We note that $u\,e^{-u^2/2}$ is of the same form as $x\,e^{-x^2/2}$

Thus $x\,e^{-x^2/2}$ **is self reciprocal under the Fourier sine transform.**

---

21. *Find the inverse Fourier sine transform of* $\hat{f}_s(\alpha) = \frac{1}{\alpha}e^{-a\alpha}$, $a > 0$

>> By data $\hat{f}_s(\alpha) = \dfrac{e^{-a\alpha}}{\alpha}$

i.e., $$F_s[f(x)] = \int_0^\infty f(x)\sin\alpha x\,dx = \hat{f}_s(\alpha) \text{ and hence}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \hat{f}_s(\alpha)\sin\alpha x\,d\alpha,$$ being the inverse Fourier sine transform.

i.e., $$f(x) = \frac{2}{\pi}\int_0^\infty \frac{e^{-a\alpha}}{\alpha}\sin\alpha x\,d\alpha = \frac{2}{\pi}\,\phi(x) \quad (say)$$

where $\phi(x) = \displaystyle\int_0^\infty \frac{e^{-a\alpha}}{\alpha}\sin\alpha x\,d\alpha$ and we need to differentiate under the integral sign for evaluation.

$$\therefore \quad \phi'(x) = \int_0^\infty \frac{e^{-a\alpha}}{\alpha}\cos\alpha x\cdot\alpha\,d\alpha = \int_0^\infty e^{-a\alpha}\cos\alpha x\,d\alpha$$

$$\phi'(x) = \left[\frac{e^{-a\alpha}}{a^2+x^2}(-a\cos\alpha x + x\sin\alpha x)\right]_{\alpha=0}^{\infty}$$ by a standard formula.

$$\phi'(x) = \frac{1}{a^2+x^2} \cdot 0 - (-a) = \frac{a}{a^2+x^2}$$

$\therefore \quad \phi(x) = \int \frac{a}{a^2+x^2}\, dx + c$

i.e., $\quad \phi(x) = \tan^{-1}(x/a) + c$

To evaluate c let us put $x = 0$.

$$\phi(0) = \tan^{-1}(0) + c \text{ or } 0 = 0 + c \quad \therefore \; c = 0.$$

Hence $\phi(x) = \tan^{-1}(x/a)$ and we have, $f(x) = \frac{2}{\pi}\phi(x)$

Thus $\quad \boldsymbol{f(x) = \frac{2}{\pi}\tan^{-1}(x/a)}$

---

22. *Find the Fourier cosine transform of* $f(x) = \frac{1}{1+x^2}$

>> By the definition, $F_c[f(x)] = \int_0^\infty f(x)\cos ux\, dx = F_c(u)$

i.e., $\quad F_c(u) = \int_0^\infty \frac{1}{1+x^2}\cos ux\, dx \quad \ldots (1)$

We cannot evaluate the R.H.S. directly and hence we employ Leibnitz rule for differentiation under the integral sign.

$$\frac{dF_c(u)}{du} = F_c'(u) = \int_0^\infty \frac{1}{1+x^2}(-\sin ux)\, x\, dx$$

i.e., $\quad F_c'(u) = -\int_0^\infty \frac{x}{1+x^2}\sin ux\, dx$

(*We cannot evaluate R.H.S even now and hence modify the integrand*)

$$F_c'(u) = -\int_0^\infty \frac{x^2}{x(1+x^2)}\sin ux\, dx$$

$$= -\int_0^\infty \frac{(1+x^2)-1}{x(1+x^2)}\sin ux\, dx$$

$$F_c'(u) = -\int_0^\infty \frac{\sin ux}{x}\,dx + \int_0^\infty \frac{\sin ux}{x(1+x^2)}\,dx$$

We note that $\int_0^\infty \frac{\sin ux}{x}\,dx = \frac{\pi}{2}$ and hence we have

$$F_c'(u) = -\frac{\pi}{2} + \int_0^\infty \frac{\sin ux}{x(1+x^2)}\,dx \qquad \ldots(2)$$

Differentiating again by Leibnitz rule we get

$$F_c''(u) = \int_0^\infty \frac{\cos ux \cdot x}{x(1+x^2)}\,dx = \int_0^\infty \frac{\cos ux}{1+x^2}\,dx$$

$\therefore \quad F_c''(u) = F_c(u)$, by using (1).

or $\quad F_c''(u) - F_c(u) = 0$ and this is a second order D.E. of the form

$$\left(D^2 - 1\right)F_c(u) = 0, \text{ where } D = \frac{d}{du}$$

A.E is $m^2 - 1 = 0 \;\therefore\; m = 1, -1$

The general solution is given by

$$F_c(u) = c_1 e^u + c_2 e^{-u}$$

We shall find c_1 and c_2.

We have $F_c(0) = c_1 + c_2$ from (3).

But from (1), $F_c(0) = \int_0^\infty \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_0^\infty = \frac{\pi}{2}$

So we have, $c_1 + c_2 = \pi/2 \qquad \ldots(4)$

Also from (3), $F_c'(u) = c_1 e^u - c_2 e^{-u}$ and hence $F_c'(0) = c_1 - c_2$

From (2), $F_c'(0) = -\pi/2 + 0 = -\pi/2$

So, we also have, $c_1 - c_2 = \pi/2 \qquad \ldots(5)$

By solving (4) and (5) we get $c_1 = 0$ and $c_2 = \pi/2$

Using these values in (3) we have $F_c(u) = (\pi/2)\, e^{-u}$

Thus the required $F_c\left[\dfrac{1}{1+x^2}\right] = \dfrac{\pi}{2} e^{-u}$

---

23. *Find the Fourier sine transform of* $f(x) = \dfrac{1}{x(1+x^2)}$

>> $\quad F_s[f(x)] = \displaystyle\int_0^\infty f(x) \sin ux \, dx = F_s(u)$

(*This problem is similar to the previous one*)

i.e., $\quad F_s(u) = \displaystyle\int_0^\infty \frac{\sin ux}{x(1+x^2)}\, dx \qquad \ldots(1)$

We cannot evaluate the integral directly and hence employ Leibnitz rule for differentiation under the integral sign.

$\therefore \quad F_s'(u) = \displaystyle\int_0^\infty \frac{x \cos ux}{x(1+x^2)}\, dx = \int_0^\infty \frac{\cos ux}{(1+x^2)}\, dx \qquad \ldots(2)$

Differentiating again w. r. t. u,

$$F_s''(u) = \int_0^\infty \frac{-x \sin ux}{1+x^2}\, dx = -\int_0^\infty \frac{x^2 \sin ux}{x(1+x^2)}\, dx$$

ie., $\quad F_s''(u) = -\displaystyle\int_0^\infty \frac{(1+x^2)-1}{x(1+x^2)} \sin ux \, dx$

$$F_s''(u) = -\int_0^\infty \frac{\sin ux}{x}\, dx + \int_0^\infty \frac{\sin ux}{x(1+x^2)}\, dx$$

i.e., $\quad F_s''(u) = -\dfrac{\pi}{2} + F_s(u)$

or $\quad F_s''(u) - F_s(u) = \dfrac{-\pi}{2}$

i.e., $\quad \left[D^2 - 1\right] F_s(u) = -\pi/2, \;$ where $D = \dfrac{d}{du}$

A.E is $m^2 - 1 = 0 \;\therefore\; m = \pm 1$

C.F is given by $c_1 e^{u} + c_2 e^{-u}$

Also P.I $= \dfrac{-\pi/2}{D^2-1} = \dfrac{-\pi/2}{-1} = \dfrac{\pi}{2}$

The general solution is given by C.F + P.I

i.e., $F_s(u) = c_1 e^{u} + c_2 e^{-u} + \pi/2$ ... (3)

We need to find c_1 and c_2

$$F_s'(u) = c_1 e^{u} - c_2 e^{-u} \qquad \ldots (4)$$

Putting $u = 0$ in (3) and (4) we have

$$F_s(0) = c_1 + c_2 + \pi/2 \text{ and } F_s'(0) = c_1 - c_2$$

But $F_s(0) = 0$ from (1) and $F_s'(0) = \int_0^\infty \dfrac{1}{1+x^2}\,dx$ from (2).

i.e., $F_s'(0) = \left[\tan^{-1} x\right]_0^\infty = \pi/2$

We have the system of equations

$$c_1 + c_2 + \pi/2 = 0 \quad \text{and} \quad c_1 - c_2 = \pi/2$$

By solving we get $c_1 = 0$, $c_2 = -\pi/2$ and we substitute these values in (3).

Thus $\boldsymbol{F_s(u) = \pi/2 \cdot (1 - e^{-u})}$

---

24. *Find the function $f(x)$ whose Fourier cosine transform is given by*

$$F(\alpha) = \begin{cases} a - (\alpha/2), & 0 \le \alpha < 2a \\ 0 & , \alpha > 2a \end{cases}$$

>> By the definition $f(x) = \dfrac{2}{\pi} \int_0^\infty F(\alpha) \cos \alpha x \, d\alpha$

$$\therefore \quad f(x) = \frac{2}{\pi} \int_0^{2a} \left(a - \frac{\alpha}{2}\right) \cos \alpha x \, d\alpha.$$

Applying Bernoulli's rule,

$$f(x) = \frac{2}{\pi}\left[\left(a - \frac{\alpha}{2}\right)\frac{\sin \alpha x}{x} - \left(\frac{-1}{2}\right)\left(\frac{-\cos \alpha x}{x^2}\right)\right]_{\alpha = 0}^{2a}$$

$$= \frac{2}{\pi}\left[(0-0) - \frac{1}{2x^2}(\cos 2ax - 1)\right]$$

Thus $\quad \boldsymbol{f(x) = \frac{1}{\pi x^2}(1 - \cos 2ax) = \frac{2\sin^2 ax}{\pi x^2}}$

---

25. *Solve the integral equation*

$$\int_0^\infty f(x)\cos \alpha x \, dx = e^{-a\alpha}$$

>> We have by inverse Fourier cosine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F(\alpha)\cos \alpha x \, d\alpha \quad \text{and} \quad F(\alpha) = e^{-a\alpha} \text{ by data.}$$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty e^{-a\alpha}\cos \alpha x \, d\alpha$$

$$= \frac{2}{\pi}\left[\frac{e^{-a\alpha}}{(-a)^2 + x^2}(-a\cos \alpha x + x \sin \alpha x)\right]_{\alpha = 0}^{\infty}$$

Thus $\quad \boldsymbol{f(x) = \frac{2a}{\pi(a^2 + x^2)}}$

---

26. *Find the Fourier cosine transform of* e^{-ax} *and hence deduce the Fourier cosine transform of* xe^{-ax}. *Further evaluate* $\int_0^\infty \frac{\cos \lambda x}{x^2 + a^2}\, dx$

>> We have $F_c\left[e^{-ax}\right] = \frac{a}{a^2 + u^2}$ *(Refer Problem - 12)*

$$\text{i.e.,} \quad \int_0^\infty e^{-ax}\cos ux \, dx = \frac{a}{a^2 + u^2} \qquad \ldots (1)$$

Differentiating (i) w. r. t a on both sides we get,

$$\int_0^\infty e^{-ax}(-x)\cos ux\ dx = \frac{(a^2+u^2)(1)-2a^2}{(a^2+u^2)^2} = \frac{u^2-a^2}{(a^2+u^2)^2}$$

or $$\int_0^\infty (x e^{-ax})\cos ux\ dx = \frac{a^2-u^2}{(a^2+u^2)^2}$$

That is $F_c\left[x e^{-ax}\right] = \dfrac{a^2-u^2}{(a^2+u^2)^2}$

Further $F_c\left[e^{-ax}\right] = \dfrac{a}{a^2+u^2}$

$\Rightarrow$ $$e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+u^2}\cos ux\ du\,,$$ *by inverse cosine transform.*

or $$\int_0^\infty \frac{\cos ux}{a^2+u^2}\,du = \frac{\pi}{2a}e^{-ax}$$

Changing x to λ and u to x we have

$$\int_0^\infty \frac{\cos\lambda x}{a^2+x^2}\,dx = \frac{\pi}{2a}e^{-a\lambda}$$

---

27. *Find th Fourier transform of* $f(x) = \begin{cases} e^{-x}, & x>0 \\ -e^{x}, & x<0 \end{cases}$

>> $$F[f(x)] = \int_{-\infty}^{\infty} f(x)\,e^{iux}\,dx$$

$$= \int_{-\infty}^{0} -e^{x}e^{iux}\,dx + \int_0^\infty e^{-x}e^{iux}\,dx$$

$$= \int_{-\infty}^{0} -e^{(1+iu)x}\,dx + \int_0^\infty e^{-(1-iu)x}\,dx$$

$$= -\left[\frac{e^{(1+iu)x}}{1+iu}\right]_{-\infty}^{0} + \left[\frac{e^{-(1-iu)x}}{-(1-iu)}\right]_0^\infty$$

$$= \frac{-1}{1+iu} + \frac{1}{1-iu} = \frac{-1+iu+1+iu}{1+u^2}$$

Thus $$F[f(x)] = \frac{2iu}{1+u^2}$$

---

28. *Find the Fourier sine and cosine transforms of* $2e^{-3x} + 3e^{-2x}$

>> $$F_s[f(x)] = \int_0^\infty f(x)\sin ux\,dx$$

$$= 2\int_0^\infty e^{-3x}\sin ux\,dx + 3\int_0^\infty e^{-2x}\sin ux\,dx$$

$$F_s[f(x)] = 2\left[\frac{e^{-3x}}{9+u^2}(-3\sin ux - u\cos ux)\right]_0^\infty + 3\left[\frac{e^{-2x}}{4+u^2}(-2\sin ux - u\cos ux)\right]_0^\infty$$

Thus $$F_s[f(x)] = \frac{2u}{9+u^2} + \frac{3u}{4+u^2} = u\left[\frac{2}{9+u^2} + \frac{3}{4+u^2}\right]$$

$$F_c[f(x)] = 2\int_0^\infty e^{-3x}\cos ux\,dx + 3\int_0^\infty e^{-2x}\cos ux\,dx$$

$$F_c[f(x)] = 2\left[\frac{e^{-3x}}{9+u^2}(-3\cos ux + u\sin ux)\right]_0^\infty + 3\left[\frac{e^{-2x}}{4+u^2}(-2\cos ux + u\sin ux)\right]_0^\infty$$

Thus $$F_c[u] = \frac{6}{9+u^2} + \frac{6}{4+u^2} = 6\left[\frac{1}{9+u^2} + \frac{1}{4+u^2}\right]$$

---

29. *Solve the following integral equation :*

$$\int_0^\infty f(x)\sin\alpha x\,dx = \begin{cases} 10, & 0 \le \alpha < 1 \\ 20, & 1 \le \alpha < 2 \\ 0, & \alpha > 2 \end{cases}$$

>> If $\int_0^\infty f(x)\sin\alpha x\,dx = F_s(\alpha)$ then we have to find $f(x)$ where,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\alpha)\sin\alpha x\,d\alpha$$

0

$$f(x) = \frac{2}{\pi}\left[\int_0^1 10 \sin \alpha x \, d\alpha + \int_1^2 20 \sin \alpha x \, d\alpha + \int_2^\infty 0 \cdot \sin \alpha x \, d\alpha\right]$$

$$= \frac{2}{\pi}\left\{\left[\frac{-10 \cos \alpha x}{x}\right]_0^1 + \left[\frac{-20 \cos \alpha x}{x}\right]_1^2\right\}$$

$$= \frac{-20}{\pi x}\left\{\left[\cos \alpha x\right]_0^1 + 2\left[\cos \alpha x\right]_1^2\right\}$$

$$= \frac{-20}{\pi x}(\cos x - 1 + 2\cos 2x - 2\cos x) = \frac{-20}{\pi x}(-1 - \cos x + 2\cos 2x)$$

Thus $\quad \boldsymbol{f(x) = \dfrac{20}{\pi x}(1 + \cos x - 2\cos 2x)}$

---

30. *Obtain the Fourier sine and cosine transforms of* x^{n-1} *and hence show that the Fourier sine and cosine transforms of* $1/\sqrt{x}$ *are the same.*

>> Let $f(x) = x^{n-1}$

$$F_s[f(x)] = F_s(u) = \int_0^\infty x^{n-1} \sin ux \, dx \qquad \ldots (1)$$

$$F_c[f(x)] = F_c(u) = \int_0^\infty x^{n-1} \cos ux \, dx \qquad \ldots (2)$$

Consider (2) $- i \times$ (1). That is

$$F_c(u) - i F_s(u) = \int_0^\infty x^{n-1}(\cos ux - i \sin ux)\, dx$$

$$= \int_0^\infty x^{n-1} e^{-iux}\, dx$$

Put $\quad iux = t \quad \therefore \quad dx = dt/iu\,;\ t$ also varies from 0 to ∞.

$$F_c(u) - i\,F_s(u) = \int_{t=0}^{\infty} \left(\frac{t}{iu}\right)^{n-1} e^{-t}\, \frac{dt}{iu}$$

$$= \frac{1}{i^n u^n} \int_0^{\infty} e^{-t}\, t^{n-1}\, dt$$

ie., $$F_c(u) - i\,F_s(u) = \frac{\Gamma(n)}{i^n u^n} = \frac{\Gamma(n)}{u^n}\, i^{-n}$$

We shall write $i = \cos(\pi/2) + i \sin(\pi/2) = e^{i\pi/2}$

$\therefore \quad i^{-n} = e^{-i n\pi/2} = \cos(n\pi/2) - i \sin(n\pi/2)$

Hence, $F_c(u) - i\,F_s(u) = \dfrac{\Gamma(n)}{u^n}\left[\cos(n\pi/2) - i\sin(n\pi/2)\right]$

$\Rightarrow \quad \boldsymbol{F_c(u) = F_c\left[x^{n-1}\right] = \dfrac{\Gamma(n)}{u^n}\cos(n\pi/2)}$

and $\quad \boldsymbol{F_s(u) = F_s\left[x^{n-1}\right] = \dfrac{\Gamma(n)}{u^n}\sin(n\pi/2)}$

Now by taking $n = 1/2$ we obtain

$$F_c\left[\frac{1}{\sqrt{x}}\right] = \frac{\Gamma(1/2)}{u^{1/2}}\cos(\pi/4) \text{ and } F_s\left[\frac{1}{\sqrt{x}}\right] = \frac{\Gamma(1/2)}{u^{1/2}}\sin(\pi/4)$$

But $\quad \Gamma(1/2) = \sqrt{\pi}$ and $\cos(\pi/4) = 1/\sqrt{2} = \sin(\pi/4)$,

Thus $\quad \boldsymbol{F_c\left[\dfrac{1}{\sqrt{x}}\right] = \sqrt{\pi/2u} = F_s\left[\dfrac{1}{\sqrt{x}}\right]}$

EXERCISES

Find the Fourier transform of the following functions

1. $f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases}$

2. $f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| > a \end{cases}$ and hence show that $\displaystyle\int_0^{\infty} \frac{\sin x - x\cos x}{x^3}\, dx = \frac{\pi}{4}$

3. $f(x) = \begin{cases} 1+x, & -1 < x \le 0 \\ 1-x, & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$

4. $f(x) = e^{-4x^2}$

5. $f(x) = x e^{-a|x|}$

Find the Fourier sine and cosine transform of the following functions

6. $f(x) = e^{-2x}$

7. $f(x) = x e^{-2x}$

8. $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

Solve the following integral equations

9. $\int_0^\infty f(x) \sin \alpha x \, dx = \begin{cases} 1-\alpha, & 0 \le \alpha < 1 \\ 0, & \alpha > 1 \end{cases}$

10. $\int_0^\infty f(x) \sin \alpha x \, dx = \begin{cases} 1, & 0 < \alpha < 1 \\ 2, & 1 < \alpha < 2 \\ 0, & \alpha > 2 \end{cases}$

ANSWERS

1. $\dfrac{\sin au - au \cos au}{u^2}$

2. $\dfrac{2}{u^3} (\sin au - au \cos au)$

3. $\dfrac{\sin^2 (u/2)}{(u/2)}$

4. $(\sqrt{\pi}/2)\, e^{-u^2/16}$

5. $\dfrac{4iau}{(a^2 + u^2)^2}$

6. $\dfrac{u}{u^2+4}$ and $\dfrac{2}{u^2+4}$

7. $\dfrac{4u}{(u^2+4)^2}$ and $\dfrac{4-u^2}{(u^2+4)^2}$

8. $\dfrac{\sin u \sin^2 (u/2)}{(u/2)^2}$ and $\dfrac{\cos u \sin^2 (u/2)}{(u/2)^2}$

9. $\dfrac{2(x - \sin x)}{\pi x^2}$

10. $\dfrac{2}{\pi x} (1 + \cos x - 2 \cos 2x)$

PART A

Unit - III

Applications of Partial Differential Equations

3.1 Introduction

A number of problems in science and engineering will lead us to partial differential equations. In this unit we focus our attention on one dimensional wave equation, one dimensional heat equation and two dimensional Laplace's equation.

We are already familiar with the solution of homogeneous p.d.e of first and second order by the method of separation of variables. The earlier mentioned three equations are p.d.es of second order and we first discuss the various possible solutions of these equations by the method of separation of variables. Later we discuss the solution of these equations subject to a given set of boundary conditions referred to as *Boundary Value Problems (B.V.P)*

In the process of solving many B.V.Ps we will also be using the earlier discussed concept of half range Fourier series.

Finally we discuss the D'Alembert's solution of one dimensional wave equation.

3.2 Various possible solutions of standard p.d.es by the method of separation of variables.

While solving problems involving second order p.d.es by the method of separation of variables [Unit-IV, Vol-II], we assumed the roots of the Auxilary Equation (A.E.) to be real and distinct in the course of solving the associated O.D.Es. But the roots can also be zero / non zero coincident or complex. Writing the solution of the O.D.Es for all the cases will give rise to various possible solutions.

We discuss various possible solutions of the one dimensional wave equation, one dimensional heat equation and two dimensional Laplace's equation.

Referring to the working procedure in the method of separation of variables, at a particular step we equate each side comprising ODEs to a common constant k.

We need to obtain the solution of the ODEs by taking the constant k equal to

(i) zero (ii) positive : $k = +p^2$ (*say*) (iii) negative : $k = -p^2$ (*say*)

Thus we obtain three possible solutions for the associated p.d.e.

- -

3.3 Various possible solutions of the one dimensional wave equation $u_{tt} = c^2 u_{xx}$ by the method of separation of variables.

Consider $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

Let $u = XT$ where $X = X(x)$, $T = T(t)$ be the solution of the PDE.

Hence the PDE becomes

$$\frac{\partial^2 (XT)}{\partial t^2} = c^2 \frac{\partial^2 (XT)}{\partial x^2} \quad \text{or} \quad X \frac{d^2 T}{dt^2} = c^2\, T \frac{d^2 X}{dx^2}$$

Dividing by $c^2\, XT$ we have, $\dfrac{1}{c^2 T} \dfrac{d^2 T}{dt^2} = \dfrac{1}{X} \dfrac{d^2 X}{dx^2}$

Equating both sides to a common constant k we have,

$$\frac{1}{X} \frac{d^2 X}{dx^2} = k \quad \text{and} \quad \frac{1}{c^2 T} \frac{d^2 T}{dt^2} = k$$

ie., $\dfrac{d^2 X}{dx^2} - kX = 0$ and $\dfrac{d^2 T}{dt^2} - c^2 kT = 0$

or $(D^2 - k)\, X = 0$ and $(D^2 - c^2 k)\, T = 0$

where $D^2 = \dfrac{d^2}{dx^2}$ in the first equation and $D^2 = \dfrac{d^2}{dt^2}$ in the second equation.

Case (i) : Let $k = 0$

The equations become

$$D^2 X = 0 \quad \text{and} \quad D^2 T = 0$$

In both the equations A.E is $m^2 = 0$ $\therefore$ $m = 0,\ 0$

Solutions are giving by

$$X = (c_1 + c_2 x)\, e^{0x} \quad \text{and} \quad T = (c_3 + c_4 t)\, e^{0t}$$

ie., $X = (c_1 + c_2 x)$ and $T = (c_3 + c_4 t)$

Hence the solution of the PDE (*when constant is* 0) is given by

$$u = XT = (c_1 + c_2 x)\,(c_3 + c_4 t)$$

Case (ii) : Let k be positive say, $k = +p^2$

The equations become

$$(D^2 - p^2)\ X = 0 \quad \text{and} \quad (D^2 - c^2 p^2)\ T = 0$$

A.Es are $m^2 - p^2 = 0$ and $m^2 - c^2 p^2 = 0$

$\therefore \quad m^2 = p^2$ or $m = \pm\, p$ and $m^2 = c^2 p^2$ or $m = \pm\, cp$

Solutions are given by,

$$X = c_1 e^{px} + c_2 e^{-px} \quad \text{and} \quad T = c_3 e^{cpt} + c_4 e^{-cpt}$$

Hence the solution of the PDE (*when constant is positive*) is given by

$$u = XT = (c_1' e^{px} + c_2' e^{-px})\ (c_3' e^{cpt} + c_4' e^{-cpt})$$

Case (iii) : Let k be negative, say $k = -p^2$

The equations become,

$$(D^2 + p^2)\ X = 0 \quad \text{and} \quad (D^2 + c^2 p^2)\ T = 0$$

A.Es are $m^2 + p^2 = 0$ and $m^2 + c^2 p^2 = 0$

or $\quad m^2 = -p^2$ and $m^2 = -c^2 p^2$

$\therefore \quad m = \pm\, ip$ and $m = \pm\, icp$

Solutions are given by

$$X = c_1 \cos px + c_2 \sin px \quad \text{and} \quad T = c_3 \cos cpt + c_4 \sin cpt$$

Hence the solution of the PDE (*when constant is negative*) is given by

$$u = XT = (c_1'' \cos px + c_2'' \sin px)\ (c_3'' \cos cpt + c_4'' \sin cpt)$$

Remark : ***Befitting solution***

Of the three possible solutions, the solution obtained in the case (iii) is considered as the befitting / suitable solution to solve a B.V.P connected with the one dimensional wave equation as the solution involves periodic functions.

The befitting solution of the one dimensional wave equation for solving B.V.Ps is given by

$$u(x, t) = (A \cos px + B \sin px)\ (C \cos cpt + D \sin cpt)$$

3.4 Various possible solutions of the one dimensional heat equation $u_t = c^2 u_{xx}$ by the method of separation of variables.

Consider $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

Let $u = XT$ where $X = X(x)$, $T = T(t)$ be the solution of the PDE.

Hence the PDE becomes

$$\frac{\partial (XT)}{\partial t} = c^2 \frac{\partial^2 (XT)}{\partial x^2}$$

or $\quad X \dfrac{dT}{dt} = c^2 T \dfrac{d^2 X}{dx^2}$ and dividing by XT we have,

$$\frac{1}{c^2 T} \frac{dT}{dt} = \frac{1}{X} \frac{d^2 X}{dx^2}$$

Equating both sides to a common constant k we have,

$$\frac{1}{c^2 T} \frac{dT}{dt} = k \quad \text{and} \quad \frac{1}{X} \frac{d^2 X}{dx^2} = k$$

ie., $\quad \dfrac{dT}{dt} - c^2 kT = 0 \quad$ and $\quad \dfrac{d^2 X}{dx^2} - kX = 0$

or $\quad (D - c^2 k)\, T = 0 \quad$ and $\quad (D^2 - k)\, X = 0$

where $D = \dfrac{d}{dt}$ in the first equation and $D = \dfrac{d}{dx}$ in the second equation.

Case (i) : Let $k = 0$

A.Es are $m = 0$ and $m^2 = 0$. $m = 0$ and $m = 0, 0$ are the roots.

Solutions are given by

$$T = c_1 e^{0t} = c_1 \quad \text{and} \quad X = (c_2 x + c_3)\, e^{0x} = (c_2 x + c_3)$$

Hence the solution of the PDE is given by

$$u = XT = c_1 (c_2 x + c_3)$$

or $\quad \boldsymbol{u(x, t) = Ax + B}$ where $c_1 c_2 = A$ and $c_1 c_3 = B$

Case (ii) : Let k be positive say $k = +p^2$

A.Es are $m - c^2 p^2 = 0$ and $m^2 - p^2 = 0$

$\therefore \quad m = c^2 p^2$ and $m = \pm p$

Solutions are given by

$$T = c_1' e^{c^2 p^2 t} \quad \text{and} \quad X = c_2' e^{px} + c_3' e^{-px}$$

Hence the solution of the PDE is given by

$$u = XT = c_1' e^{c^2 p^2 t} \cdot (c_2' e^{px} + c_3' e^{-px})$$

or $\quad \boldsymbol{u(x, t) = e^{c^2 p^2 t} (A' e^{px} + B' e^{-px})}$ where $c_1' c_2' = A'$ and $c_1' c_3' = B'$

Case (iii) : Let k be negative, say $k = -p^2$

A.Es are $m + c^2 p^2 = 0$ and $m^2 + p^2 = 0$

$\therefore \quad m = -c^2 p^2$ and $m = \pm ip$

Solutions are given by

$$T = c_1'' e^{-c^2 p^2 t} \quad \text{and} \quad X = c_2'' \cos px + c_3'' \sin px$$

Hence the solution of the PDE is given by

$$u = XT = c_1'' e^{-c^2 p^2 t} \cdot (c_2'' \cos px + c_3'' \sin px)$$

or $\quad \boldsymbol{u(x, t) = e^{-c^2 p^2 t}} (A'' \cos px + B'' \sin px)$

where $c_1'' c_2'' = A''$ & $c_1'' c_3'' = B''$

Remark : *As remarked earlier in the case of one dimensional wave equation, the befitting / suitable solution of the one dimensional heat equation for solving B.V.Ps is given by*

$$\boldsymbol{u(x, t) = e^{-c^2 p^2 t} (A \cos px + B \sin px)}$$

3.5 Various possible solutions of the two dimensional Laplace's equation $u_{xx} + u_{yy} = 0$ by the method of separation of variables

Consider $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Let $u = XY$ where $X = X(x)$, $Y = Y(y)$ be the solution of the PDE.

Hence the PDE becomes

$$\frac{\partial^2 (XY)}{\partial x^2} + \frac{\partial^2 (XY)}{\partial y^2} = 0$$

or $\quad Y \dfrac{d^2 X}{dx^2} + X \dfrac{d^2 Y}{dy^2} = 0$ and dividing by XY we have,

$$\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{-1}{Y} \frac{d^2 Y}{dy^2}$$

Equating both sides to a common constant k we have,

$$\frac{1}{X} \frac{d^2 X}{dx^2} = k \quad \text{and} \quad \frac{-1}{Y} \frac{d^2 Y}{dy^2} = k$$

or $\quad (D^2 - k)\, X = 0$ and $(D^2 + k)\, Y = 0$

where $D = \dfrac{d}{dx}$ in the first equation and $D = \dfrac{d}{dy}$ in the second equation.

Case (i) : Let $k = 0$

A.E is $m^2 = 0$ in respect of both the equations.

$\therefore \quad m = 0, 0$ and $m = 0, 0$

Solutions are given by

$$X = c_1 x + c_2 \quad \text{and} \quad Y = c_3 y + c_4$$

Hence the solution of the PDE is given by

$$u = XY = (c_1 x + c_2)(c_3 y + c_4)$$

Case (ii) : Let k be positive, say $k = +p^2$

A.Es are $m^2 - p^2 = 0$ and $m^2 + p^2 = 0$

$\therefore \quad m = \pm p$ and $m = \pm ip$

Solutions are given by

$$X = c_1' e^{px} + c_2' e^{-px} \quad \text{and} \quad Y = c_3' \cos py + c_4' \sin py$$

Hence the solution of the PDE is given by

$$u = XY = (c_1' e^{px} + c_2' e^{-px})(c_3' \cos py + c_4' \sin py)$$

Case (iii) : Let k be negative, say $k = -p^2$

A.Es are $m^2 + p^2 = 0$ and $m^2 - p^2 = 0$

$\therefore \quad m = \pm ip$ and $m = \pm p$

Solutions are given by

$$X = c_1'' \cos px + c_2'' \sin px \quad \text{and} \quad Y = c_3'' e^{py} + c_4'' e^{-py}$$

Hence the solution of the PDE is given by

$$u = XY = (c_1'' \cos px + c_2'' \sin px)(c_3'' e^{py} + c_4'' e^{-py})$$

Remark : *We have to choose the befitting / suitable solution which conforms to the given boundary conditions in the course of solving a B.V.P associated with the Laplace's equation in two dimensions. Usually the solution obtained in the case (iii) will be befitting as it invloves periodic functions of x*

Working procedure for problems (Solving a B.V.P.)

- We assume the befitting / suitable solution associated with the three partial differential equations. (*Refer to the earlier given remarks*)
- We apply the given conditions and determine the arbitrary constants present in the solution.
- The concept of half range Fourier series will help in determining some of the constants in many problems.

WORKED PROBLEMS

Problems on Wave Equation

1. *Solve the wave equation* $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ *given that* $u(0, t) = 0$, $u(l, t) = 0$, $\dfrac{\partial u}{\partial t} = 0$ *when* $t = 0$ *and* $u(x, 0) = u_0 \sin(\pi x/l)$

>> The befitting solution for solving the given problem is represented by

$$u(x, t) = (A \cos px + B \sin px)(C \cos cpt + D \sin cpt) \quad \ldots (1)$$

Consider $u(0, t) = 0$. Now (1) becomes

$$0 = (A)(C \cos cpt + D \sin cpt)$$

This clearly implies that we must have $A = 0$

Consider $u(l, t) = 0$. Using $A = 0$, (1) becomes

$$0 = (B \sin pl)(C \cos cpt + D \sin cpt)$$

Since $A = 0$, B cannot be zero. (*If* $B = 0$ *then* $u(x, t) = 0$)

Hence we must have $\sin pl = 0$

Noting that $\sin n\pi = 0$ for $n = 1, 2, 3, \cdots$ we must have, $pl = n\pi$ or $p = n\pi/l$

Using $A = 0$ and $p = n\pi/l$ in (1) we have,

$$u(x, t) = B \sin \frac{n\pi x}{l} \left(C \cos \frac{n\pi ct}{l} + D \sin \frac{n\pi ct}{l} \right) \qquad \ldots (2)$$

Differentiating partially w.r.t t we have,

$$\frac{\partial u}{\partial t} = B \sin \frac{n\pi x}{l} \left(-C \sin \frac{n\pi ct}{l} + D \cos \frac{n\pi ct}{l} \right) \cdot \left(\frac{n\pi c}{l} \right)$$

Consider $\dfrac{\partial u}{\partial t} = 0$ when $t = 0$

$$\therefore \qquad 0 = B \sin \frac{n\pi x}{l} (D) \left(\frac{n\pi c}{l} \right)$$

Since $B \neq 0$, we must have $D = 0$. Using $D = 0$ in (2) we have,

$$u(x, t) = BC \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}; \; n = 1, 2, 3, \cdots$$

We shall take $n = 1, 2, 3, \cdots$ and correspondingly take the constant $BC = b_1, b_2, b_3 \cdots$

In view of this, we have a set of independent solutions satisfying three of the given conditions.

It is evident that their sum also satisfies the same conditions.

On adding these independent solutions we get,

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad \ldots (3)$$

We now consider the last condition $u(x, 0) = u_0 \sin(\pi x/l)$

Putting $t = 0$ in (3) we have,

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \text{ since } \cos 0 = 1$$

ie., $$u_0 \sin \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

or $$u_0 \sin \frac{\pi x}{l} = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + \cdots$$

Comparing both sides we get, $b_1 = u_0$, $b_2 = 0$, $b_3 = 0$, $\cdots$

Thus by substituting these values in the expanded form of (3) we get,

$$u(x,t) = u_0 \sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}$$

---

2. *Solve the wave equation $u_{tt} = c^2 u_{xx}$ subject to the conditions*

$$u(0,t)=0,\ u(l,t)=0,\ \frac{\partial u}{\partial t}(x,0)=0 \text{ and } u(x,0)=u_0\sin^3(\pi x/l)$$

>> [***It may be noted that the first three conditions are same as in problem-1. It is necessary to retrace / give all the steps upto the stage of getting equation (3)***]

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad \ldots(3)$$

Consider $u(x,0) = u_0 \sin^3(\pi x/l)$. We have from (3),

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

ie., $$u_0 \sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

We know that $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

$$\therefore \quad u_0\left[\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right] = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

ie., $$\frac{3u_0}{4}\sin\frac{\pi x}{l} - \frac{u_0}{4}\sin\frac{3\pi x}{l} = b_1\sin\frac{\pi x}{l} + b_2\sin\frac{2\pi x}{l} + b_3\sin\frac{3\pi x}{l} + \cdots$$

Comparing both sides we get,

$$b_1 = 3u_0/4,\ b_2 = 0,\ b_3 = -u_0/4,\ b_4 = 0,\ b_5 = 0,\cdots$$

Thus by substituting these values in the expanded form of (3) we get,

$$u(x,t) = \frac{3u_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{u_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}$$

---

3. *Solve the wave equation $u_{tt} = c^2 u_{xx}$ subject to the conditions*

$$u(0,t)=0,\ u(l,t)=0,\ \frac{\partial u}{\partial t}=0 \text{ when } t=0 \text{ and } u(x,0)=f(x)$$

>> [***It may be noted that the first three conditions are same as in problem-1. It is necessary to retrace / give all the steps upto the stage of getting equation (3)***]

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad \ldots(3)$$

Consider $u(x, 0) = f(x)$. We have from (3),

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

ie., $$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

[**Note** : *In the previous two problems L.H.S was also of the form* $b_n \sin(n\pi x/l)$ *for some particular value(s) of n and hence we could easily find the constants* $b_1, b_2, b_3 \cdots$ *individually by simple comparison. In the present situation* b_n *will be presented using the concept of half range Fourier series*]

The series in R.H.S is regarded as the sine half range Fourier series of $f(x)$ in $(0, l)$ and hence

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx \qquad \ldots (4)$$

Thus we have the required solution in the form

$$\boldsymbol{u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}}$$

where b_n is given by (4).

Remark : *In the subsequent problems [problems 4&5]* $f(x)$ *will be given specifically and hence we will be finding* b_n *by carrying out the integration.*

---

4. *Solve the wave equation* $u_{tt} = c^2 u_{xx}$ *given that*
$u(0, t) = 0 = u(l, t)$, $u_t(x, 0) = 0$ *and* $u(x, 0) = x(l - x)$

>> ***[It may be noted that the first three conditions are same as in problem-1. It is necessary to retrace / give all the steps upto the stage of getting equation (3)]***

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad \ldots (3)$$

Consider $u(x, 0) = x(l - x) = (lx - x^2)$. We have from (3),

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

ie., $$(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

The series in R.H.S is regarded as the sine half range Fourier series of $(lx - x^2)$ in $(0, l)$ and hence

$$b_n = \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

[*Refer problem - 33 of unit-I for the integration process*]

$$b_n = \frac{4l^2}{n^3\pi^3}\{1 - (-1)^n\} \quad \text{or} \quad b_n = \begin{cases} 8l^2/n^3\pi^3 & \text{when } n \text{ is odd} \\ 0 & \text{when } n \text{ is even} \end{cases}$$

We substitute for b_n in (3).

Thus the required solution is given by

$$u(x, t) = \sum_{n=1,3,5\cdots}^{\infty} \frac{8l^2}{n^3\pi^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$$

---

5. *Show that the solution of the B.V.P governed by the p.d.e* $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ *subject to the conditions* $u(0, t) = 0 = u(l, t)$, $\frac{\partial u}{\partial t}(x, 0) = 0$ *and* $u(x, 0) = f(x)$ *where*

$$f(x) = \begin{cases} \dfrac{2kx}{l} & \text{in } 0 \le x \le l/2 \\ \dfrac{2k}{l}(l - x) & \text{in } l/2 \le x \le l \end{cases}$$ *is given by*

$$\frac{8k}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{1}{3^2}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l} + \cdots\right]$$

>> ***[It may be noted that the first three conditions are same as in problem-1. It is necessary to retrace / give all the steps upto the stage of getting equation (3)]***

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \quad \ldots (3)$$

Consider $u(x, 0) = f(x)$. We have from (3),

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

ie., $$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

The series in R.H.S is regarded as the sine half range Fourier series of $f(x)$ in $(0, l)$ and hence

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

Referring to the given $f(x)$ we have,

$$b_n = \frac{2}{l} \left[\int_0^{l/2} \frac{2kx}{l} \sin \frac{n\pi x}{l} \, dx + \int_{l/2}^{l} \frac{2k}{l} (l - x) \sin \frac{n\pi x}{l} \, dx \right]$$

$$= \frac{4k}{l^2} \left[\int_0^{l/2} x \sin \frac{n\pi x}{l} \, dx + \int_{l/2}^{l} (l - x) \sin \frac{n\pi x}{l} \, dx \right]$$

$$= \frac{4k}{l^2} \left\{ \left[x \cdot \frac{-\cos \frac{n\pi x}{l}}{(n\pi/l)} - 1 \cdot \frac{-\sin \frac{n\pi x}{l}}{(n\pi/l)^2} \right]_0^{l/2} \right.$$

$$\left. + \left[(l - x) \cdot \frac{-\cos \frac{n\pi x}{l}}{(n\pi/l)} - (-1) \cdot \frac{-\sin \frac{n\pi x}{l}}{(n\pi/l)^2} \right]_{l/2}^{l} \right\}$$

$$= \frac{4k}{l^2} \left\{ \frac{-l}{n\pi} \left(\frac{l}{2} \cos \frac{n\pi}{2} \right) + \frac{l^2}{n^2 \pi^2} \left(\sin \frac{n\pi}{2} \right) - \frac{l}{n\pi} \left(\frac{-l}{2} \cos \frac{n\pi}{2} \right) - \frac{l^2}{n^2 \pi^2} \left(-\sin \frac{n\pi}{2} \right) \right\}$$

Hence $b_n = \frac{4k}{l^2} \cdot \frac{2l^2}{n^2 \pi^2} \sin \frac{n\pi}{2} = \frac{8k}{n^2 \pi^2} \sin \frac{n\pi}{2}$

We substitute for b_n in (3)

$$u(x, t) = \sum_{n=1}^{\infty} \frac{8k}{n^2 \pi^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

Thus we have on expanding the R.H.S.

$$u(x, t) = \frac{8k}{\pi^2} \left[\frac{1}{1^2} \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{1}{3^2} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l} + \cdots \right]$$

---

6. *Solve the one dimensional wave equation* $u_{tt} = c^2 u_{xx}$ *subject to the conditions*

$$u(0, t) = 0, \; u(2l, t) = 0, \; u(x, 0) = 0 \;\; and \;\; \frac{\partial u}{\partial t} = a \sin \frac{\pi x}{2l} \;\; at \;\; t = 0$$

>> *[Note that the last two conditions are different in comparison with the type of conditions in problems 1 to 5]*

The befitting solution for solving the given problem is represented by

$$u(x, t) = (A\cos px + B\sin px)(C\cos cpt + D\sin cpt) \quad \ldots (1)$$

Consider $u(0, t) = 0$. Now (1) becomes

$$0 = (A)(C\cos cpt + D\sin cpt) \therefore A = 0$$

Consider $u(2l, t) = 0$. Using $A = 0$ (1) becomes

$$0 = (B\sin p \cdot 2l)(C\cos cpt + D\sin cpt)$$

Since $A = 0$, B cannot be zero. (*If* $B = 0$ *then* $u(x, t) = 0$)

Hence we must have $\sin(p \cdot 2l) = 0$ or $p \cdot 2l = n\pi \therefore p = n\pi/2l$

Using $A = 0$ and $p = n\pi/2l$ in (1) we have,

$$u(x, t) = B\sin\frac{n\pi x}{2l}\left(C\cos\frac{n\pi ct}{2l} + D\sin\frac{n\pi ct}{2l}\right) \quad \ldots (2)$$

Now consider $u(x, 0) = 0$

We have from (2), $0 = BC\sin\frac{n\pi x}{2l}$

Since $B \neq 0$, we must have $C = 0$.

Now, $u(x, t) = BD\sin\frac{n\pi x}{2l}\sin\frac{n\pi ct}{2l}$; $n = 1, 2, 3, \cdots$

We shall take $n = 1, 2, 3, \ldots$ and correspondingly take the constant $BD = b_1, b_2, b_3, \ldots$

In view of this, we have a set of independent solutions satisfying three of the given conditions. It is evident that their sum also satisfy the same conditions.

On adding these independent solutions we get,

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2l}\sin\frac{n\pi ct}{2l} \quad \ldots (3)$$

Now, $\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2l}\cos\frac{n\pi ct}{2l}\left(\frac{n\pi c}{2l}\right)$

Consider the last condition, $\frac{\partial u}{\partial t} = a\sin\frac{\pi x}{2l}$ at $t = 0$

At $t = 0$, $\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2l}\left(\frac{n\pi c}{2l}\right) = \frac{\pi c}{2l}\sum_{n=1}^{\infty}(n b_n)\sin\frac{n\pi x}{2l}$

i.e., $a \sin \dfrac{\pi x}{2l} = \dfrac{\pi c}{2l}\left[b_1 \sin \dfrac{\pi x}{2l} + 2b_2 \sin \dfrac{2\pi x}{2l} + 3b_3 \sin \dfrac{3\pi x}{2l} + \cdots \right]$

Comparing both sides we get,

$$a = \frac{\pi c}{2l} \cdot b_1 \text{ or } b_1 = \frac{2al}{\pi c} \; ; \; b_2 = 0, \; b_3 = 0, \ldots$$

Thus, by substituting these values in the expanded form of (3) we get,

$$\boldsymbol{u(x,t) = \frac{2al}{\pi c} \sin \frac{\pi x}{2l} \sin \frac{\pi c t}{2l}}$$

---

7. *Solve the wave equation* $u_{tt} = c^2 u_{xx}$ *given that* $u(0,t) = 0 = u(2l,t)$,

$u(x,0) = 0$ *and* $\dfrac{\partial u}{\partial t}(x,0) = a \sin^3 \dfrac{\pi x}{2l}$

>> [***We need to retrace all the steps of prob.-6 upto the stage of getting equation (3)***]

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2l} \sin \frac{n\pi c t}{2l} \quad \ldots (3)$$

Now, $\dfrac{\partial u}{\partial t} = \displaystyle\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2l} \cos \frac{n\pi c t}{2l}\left(\frac{n\pi c}{2l}\right)$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2l}\left(\frac{n\pi c}{2l}\right) = \frac{\pi c}{2l}\sum_{n=1}^{\infty} (n b_n) \sin \frac{n\pi x}{2l}$$

i.e., $a \sin^3 \dfrac{\pi x}{2l} = \dfrac{\pi c}{2l}\displaystyle\sum_{n=1}^{\infty} (n b_n) \sin \frac{n\pi x}{2l}$

Using $\sin^3 \theta = \dfrac{3}{4} \sin \theta - \dfrac{1}{4} \sin 3\theta$ where $\theta = \pi x/2l$ in L.H.S, we have,

$$\frac{3a}{4} \sin \frac{\pi x}{2l} - \frac{a}{4} \sin \frac{3\pi x}{2l}$$

$$= \frac{\pi c}{2l}\left[b_1 \sin \frac{\pi x}{2l} + 2b_2 \sin \frac{2\pi x}{2l} + 3b_3 \sin \frac{3\pi x}{2l} + \cdots \right]$$

$\Rightarrow$ $\dfrac{\pi c}{2l} \cdot b_1 = \dfrac{3a}{4}$ or $b_1 = \dfrac{3al}{2\pi c}$; $b_2 = 0$

$\dfrac{\pi c}{2l} \cdot 3b_3 = -\dfrac{a}{4}$ or $b_3 = -\dfrac{al}{6\pi c}$; $b_4 = 0, \; b_5 = 0 \ldots$

Thus by substituting these values in the expanded form of (3) we get,

$$u(x,t) = \frac{3al}{2\pi c}\sin\frac{\pi x}{2l}\sin\frac{\pi c t}{2l} - \frac{al}{6\pi c}\sin\frac{3\pi x}{2l}\sin\frac{3\pi c t}{2l}$$

---

8. *Solve the B.V.P governed by the p.d.e* $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ *subjected to the conditions,*
$u(0,t) = 0, u(2l,t) = 0, u(x,0) = 0$ *and*

$$\frac{\partial u}{\partial t}(x,0) = v(x) \text{ where } v(x) = \begin{cases} \frac{x}{l} & in\ 0 \le x \le l \\ \frac{2l-x}{l} & in\ l \le x \le 2l \end{cases}$$

>> **[*We need to retrace all the steps of prob.-6 upto the stage of getting equation (3)*]**

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2l}\sin\frac{n\pi c t}{2l} \qquad \ldots(3)$$

Now, $\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2l}\cos\frac{n\pi ct}{2l}\cdot\left(\frac{n\pi c}{2l}\right)$

At $\quad t = 0, \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2l}\left(\frac{n\pi c}{2l}\right)$

i.e., $\quad v(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2l}$ where $B_n = b_n\left(\frac{n\pi c}{2l}\right)$

The series in R.H.S is regarded as the sine half range Fourier series of $v(x)$ in $(0,2l)$ and hence

$$B_n = \frac{2}{2l}\int_0^{2l} f(x)\sin\frac{n\pi x}{2l}\,dx$$

$$= \frac{1}{l}\left[\int_0^{l}\frac{x}{l}\sin\frac{n\pi x}{2l}\,dx + \int_l^{2l}\frac{2l-x}{l}\sin\frac{n\pi x}{2l}\,dx\right]$$

$$= \frac{1}{l^2}\left[\int_0^{l} x\sin\frac{n\pi x}{2l}\,dx + \int_l^{2l}(2l-x)\sin\frac{n\pi x}{2l}\,dx\right]$$

$$B_n = \frac{1}{l^2}\left\{\left[x \cdot \frac{-\cos\frac{n\pi x}{2l}}{(n\pi/2l)} - (1)\frac{-\sin\frac{n\pi x}{2l}}{(n\pi/2l)^2}\right]_0^l\right.$$

$$\left. + \left[(2l-x)\frac{-\cos\frac{n\pi x}{2l}}{(n\pi/2l)} - (-1)\frac{-\sin\frac{n\pi x}{2l}}{(n\pi/2l)^2}\right]_l^{2l}\right\}$$

$$B_n = \frac{1}{l^2}\left\{-\frac{2l}{n\pi}\left(l\cos\frac{n\pi}{2}\right) + \frac{4l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)\right.$$

$$\left. - \frac{2l}{n\pi}\left(-l\cos\frac{n\pi}{2}\right) - \frac{4l^2}{n^2\pi^2}\left(-\sin\frac{n\pi}{2}\right)\right\}$$

$$B_n = \frac{1}{l^2}\cdot\frac{8l^2}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

i.e., $$b_n\left(\frac{n\pi c}{2l}\right) = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$\therefore$ $$b_n = \frac{16l}{n^3\pi^3 c}\sin\frac{n\pi}{2}$$

Thus by substituting this value in (3) we get,

$$u(x,t) = \sum_{n=1}^{\infty}\frac{16l}{n^3\pi^3 c}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\sin\frac{n\pi ct}{2l}$$

---

9. *Solve the wave equation* $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ *under the conditions* $u(0,t) = 0 = u(l,t)$ *for all t,* $u(x,0) = f(x)$ *and* $\frac{\partial u}{\partial t}(x,0) = g(x)$

>> The befitting solution for solving the problem is represented by,

$$u(x,t) = (A\cos px + B\sin px)(C\cos cpt + D\sin cpt) \quad \ldots(1)$$

Consider $u(0,t) = 0$. Now (1) becomes

$$0 = (A)(C\cos cpt + D\sin cpt) \quad \therefore \quad A = 0$$

Consider $u(l,t) = 0$. Using $A = 0$, (1) becomes

$$0 = (B\sin pl)(C\cos cpt + D\sin cpt)$$

Since $A = 0$, B cannot be zero. (*If* $B = 0$ *then* $u(x,t) = 0$)

Hence we must have $\sin pl = 0$ or $pl = n\pi$ $\therefore$ $p = n\pi/l$

Using $A = 0$ and $p = n\pi/l$ in (1) we have,

$$u(x,t) = B \sin \frac{n\pi x}{l}\left(C \cos \frac{n\pi ct}{l} + D \sin \frac{n\pi ct}{l}\right)$$

ie.,
$$u(x,t) = \sin \frac{n\pi x}{l}\left(BC \cos \frac{n\pi ct}{l} + BD \sin \frac{n\pi ct}{l}\right)$$

Taking $n = 1, 2, 3, \cdots$ and $BC = b_1, b_2, b_3, \cdots$, $BD = b_1', b_2', b_3' \cdots$ we have a set of independent solutions,

$$u(x,t) = \sin \frac{n\pi x}{l}\left(b_n \cos \frac{n\pi ct}{l} + b_n' \sin \frac{n\pi ct}{l}\right); \; n = 1, 2, 3, \cdots$$

satisfying the first two given conditions.

It is evident that their sum also satisfy the same conditions.

On adding these independent solutions we get,

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l}\left(b_n \cos \frac{n\pi ct}{l} + b_n' \sin \frac{n\pi ct}{l}\right) \quad \ldots (2)$$

Now consider the condition $u(x, 0) = f(x)$

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

ie.,
$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

The series in R.H.S is regarded as the sine half range Fourier series of $f(x)$ in $(0, l)$ and hence

$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx \quad \ldots (3)$$

Also by differentiating (2) partially w.r.t t we have,

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l}\left(-b_n \sin \frac{n\pi ct}{l} + b_n' \cos \frac{n\pi ct}{l}\right)\cdot\left(\frac{n\pi c}{l}\right)$$

We consider the last condition $\frac{\partial u}{\partial t}(x, 0) = g(x)$

$$\frac{\partial u}{\partial t}(x, 0) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \cdot b_n' \cdot \left(\frac{n\pi c}{l}\right)$$

ie., $g(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$ where $B_n = b_n' \left(\frac{n\pi c}{l} \right)$

Again, the series in R.H.S is regarded as the sine half range Fourier series of $g(x)$ in $(0, l)$ and hence

$$B_n = \frac{2}{l} \int_0^l g(x) \sin \frac{n\pi x}{l} \, dx$$

ie., $$b_n' \left(\frac{n\pi c}{l} \right) = \frac{2}{l} \int_0^l g(x) \sin \frac{n\pi x}{l} \, dx$$

$$\therefore \quad b_n' = \frac{2}{n\pi c} \int_0^l g(x) \sin \frac{n\pi x}{l} \, dx \quad \ldots (4)$$

Thus the required solution as in (2) is given by

$$u(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left(b_n \cos \frac{n\pi ct}{l} + b_n' \sin \frac{n\pi ct}{l} \right)$$

where, $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$ and $b_n' = \frac{2}{n\pi c} \int_0^l g(x) \sin \frac{n\pi x}{l} \, dx$

Note : *If $f(x)$ and $g(x)$ are given specifically, we obtain the corresponding b_n and b_n' by completing the integration process.*

Problems on Heat Equation

10. *Obtain the solution of the heat equation* $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ *subject to the conditions* $u(0, t) = 0$, $u(l, t) = 0$ *and* $u(x, 0) = f(x)$

>> The befitting solution for solving the problem is represented by

$$u(x, t) = e^{-c^2 p^2 t} (A \cos px + B \sin px) \quad \ldots (1)$$

Consider $u(0, t) = 0$. Now (1) becomes

$$0 = e^{-c^2 p^2 t} (A) \quad \therefore \; A = 0$$

Consider $u(l, t) = 0$. Using $A = 0$, (1) becomes

$$0 = e^{-c^2 p^2 t} (B \sin pl)$$

Since $A = 0$, B cannot be zero (*If* $B = 0$ *then* $u(x, t) = 0$)

Hence we must have $\sin pl = 0$ or $pl = n\pi \quad \therefore \quad p = n\pi/l$

Using $A = 0$ and $p = n\pi/l$ in (1) we have,

$$u(x, t) = e^{\frac{-n^2\pi^2c^2t}{l^2}} \left(B \sin \frac{n\pi x}{l}\right) ; \quad n = 1, 2, 3, \cdots$$

We shall take $n = 1, 2, 3, \cdots$ and correspondingly take the constant $B = b_1, b_2, b_3, \cdots$

In view of this we have a set of independent solutions satisfying the first two of the given conditions.

It is evident that their sum also satisfy the same conditions.

On adding these independent solutions we get

$$u(x, t) = \sum_{n=1}^{\infty} b_n \, e^{\frac{-n^2\pi^2c^2t}{l^2}} \sin \frac{n\pi x}{l} \qquad \ldots (2)$$

We now consider the last condition $u(x, 0) = f(x)$. We have from (2),

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

ie.,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

The series in R.H.S is regarded as the sine half range Fourier series of $f(x)$ in $(0, l)$ and hence

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx \qquad \ldots (3)$$

Thus the required solution $u(x, t)$ is given by (2) where b_n is given by (3).

---

11. *Solve the heat equation* $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ *given that* $u(0, t) = 0$, $u(l, t) = 0$ *and* $u(x, 0) = 100\,x/l$

>> [***We need to retrace all the steps of problem-10. Further we have to find the constant b_n for the given $f(x)$***]

$$b_n = \frac{2}{l} \int_0^l \frac{100\,x}{l} \sin \frac{n\pi x}{l} \, dx = \frac{200}{l^2} \int_0^l x \sin \frac{n\pi x}{l} \, dx$$

$$b_n = \frac{200}{l^2}\left[x \cdot \frac{-\cos\frac{n\pi x}{l}}{(n\pi/l)} - 1 \cdot \frac{-\sin\frac{n\pi x}{l}}{(n\pi/l)^2} \right]_0^l \text{, by Bernoulli's rule.}$$

$$b_n = \frac{200}{l^2} \cdot \frac{-l}{n\pi}(l \cos n\pi) = \frac{-200}{n\pi}(-1)^n = \frac{200(-1)^{n+1}}{n\pi}$$

The required solution is obtained by substituting this value of b_n in (2).

Thus $$u(x, t) = \sum_{n=1}^{\infty} \frac{200(-1)^{n+1}}{n\pi} e^{-\frac{n^2\pi^2c^2t}{l^2}} \sin\frac{n\pi x}{l}$$

---

12. *Obtain the solution of the heat equation* $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ *given that* $u(0, t) = 0 = u(l, t)$ *and* $u(x, 0) = f(x)$ *where,*

$$f(x) = \begin{cases} \frac{2Tx}{l} & \text{in } 0 \le x \le l/2 \\ \frac{2T}{l}(l-x) & \text{in } l/2 \le x \le l \end{cases}$$

>> [***We need to retrace all the steps of problem-10. Further we have to find the constant*** b_n ***for the given*** $f(x)$]

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$$

$$b_n = \frac{2}{l}\left[\int_0^{l/2} \frac{2Tx}{l} \sin\frac{n\pi x}{l}\, dx + \int_{l/2}^{l} \frac{2T}{l}(l-x) \sin\frac{n\pi x}{l}\, dx\right]$$

$$= \frac{4T}{l^2}\left[\int_0^{l/2} x \sin\frac{n\pi x}{l}\, dx + \int_{l/2}^{l}(l-x) \sin\frac{n\pi x}{l}\, dx\right]$$

[*Refer problem - 5 for the integration process*]

$$b_n = \frac{8T}{n^2\pi^2} \sin\frac{n\pi}{2}$$

The required solution is obtained by substituting this value of b_n in (2).

Thus $$u(x,t) = \frac{8T}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi}{2}\, e^{\frac{-n^2\pi^2c^2t}{l^2}} \sin\frac{n\pi x}{l}$$

---

13. *Solve the equation* $u_t = c^2 u_{xx}$ *given that* $u(0,t) = 0$, $u(30,t) = 0$ *and* $u(x,0) = 2x + 20$

\>> The befitting solution for solving the given problem is represented by

$$u(x,t) = e^{-c^2p^2t}(A\cos px + B\sin px) \qquad \ldots (1)$$

Consider $u(0,t) = 0$. Now (1) becomes

$$0 = e^{-c^2p^2t}(A) \quad \therefore\ A = 0$$

Consider $u(30,t) = 0$. Using $A = 0$, (1) becomes

$$0 = e^{-c^2p^2t}(B\sin 30p)$$

Since $B \neq 0$, $\sin 30p = 0$ or $30p = n\pi$ $\therefore$ $p = n\pi/30$

Now $$u(x,t) = e^{\frac{-n^2\pi^2c^2t}{900}}\left(B\sin\frac{n\pi x}{30}\right)$$

In general, $$u(x,t) = \sum_{n=1}^{\infty} b_n\, e^{\frac{-n^2\pi^2c^2t}{900}} \sin\frac{n\pi x}{30} \qquad \ldots (2)$$

Consider $u(x,0) = 2x + 20$ and we have from (2).

$$2x + 20 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{30}$$

The series in R.H.S is regarded as the sine half range Fourier series of $(2x + 20)$ in $(0, 30)$ and hence

$$b_n = \frac{2}{30}\int_0^{30}(2x + 20)\sin\frac{n\pi x}{30}\,dx$$

$$b_n = \frac{1}{15}\left[(2x+20)\cdot\frac{-\cos\frac{n\pi x}{30}}{(n\pi/30)} - 2\cdot\frac{-\sin\frac{n\pi x}{30}}{(n\pi/30)^2}\right]_0^{30}$$

$$= \frac{-2}{n\pi}\left[(2x+20)\cos\frac{n\pi x}{30}\right]_0^{30}$$

$$b_n = \frac{-2}{n\pi}\{80\cos n\pi - 20\} = \frac{40}{n\pi}\{1-4\cos n\pi\} = \frac{40}{n\pi}\{1-4(-1)^n\}$$

The required solution is obtained by substituting this value of b_n in (2)

Thus $$u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n\pi}\left\{1-4(-1)^n\right\} e^{\frac{-n^2\pi^2c^2t}{900}} \sin\frac{n\pi x}{30}$$

---

14. *Solve the heat equation* $u_t = c^2 u_{xx}$ *subject to the conditions,* $u(0,t) = 0$, $u(10,t) = 0$ *and* $u(x,0) = f(x)$ *where*

$$f(x) = \begin{cases} x & \text{in } 0 \le x \le 5 \\ 10-x & \text{in } 5 \le x \le 10 \end{cases}$$

>> The befitting solution for solving the problem is represented by

$$u(x,t) = e^{-c^2p^2t}(A\cos px + B\sin px) \qquad \ldots (1)$$

Consider $u(0,t) = 0$. Now (1) becomes

$$0 = e^{-c^2p^2t}(A) \qquad \therefore\ A = 0$$

Consider $u(10,t) = 0$. Using $A = 0$, (1) becomes

$$0 = e^{-c^2p^2t}(B\sin 10p)$$

Since $B \neq 0$, $\sin 10p = 0$ or $10p = n\pi$ $\therefore\ p = n\pi/10$

Now $$u(x,t) = e^{\frac{-n^2\pi^2c^2t}{100}}\left(B\sin\frac{n\pi x}{10}\right)$$

In general, $$u(x,t) = \sum_{n=1}^{\infty} b_n\, e^{\frac{-n^2\pi^2c^2t}{100}} \sin\frac{n\pi x}{10} \qquad \ldots (2)$$

Consider $u(x,0) = f(x)$ and we have from (2)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}$$

The series in R.H.S is regarded as the sine half range Fourier series of $f(x)$ in $(0, 10)$ and hence

$$b_n = \frac{2}{10}\int_0^{10} f(x)\sin\frac{n\pi x}{10}\,dx$$

$$b_n = \frac{1}{5}\left[\int_0^5 x\sin\frac{n\pi x}{10}\,dx + \int_5^{10}(10-x)\sin\frac{n\pi x}{10}\,dx\right]$$

$$= \frac{1}{5}\left\{\left[x\cdot\frac{-\cos\frac{n\pi x}{10}}{(n\pi/10)} - 1\cdot\frac{-\sin\frac{n\pi x}{10}}{(n\pi/10)^2}\right]_0^5 + \left[(10-x)\cdot\frac{-\cos\frac{n\pi x}{10}}{(n\pi/10)} - (-1)\cdot\frac{-\sin\frac{n\pi x}{10}}{(n\pi/10)^2}\right]_5^{10}\right\}$$

$$= \frac{1}{5}\left[\frac{-10}{n\pi}\left(5\cos\frac{n\pi}{2}\right) + \frac{100}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right) - \frac{10}{n\pi}\left(-5\cos\frac{n\pi}{2}\right) - \frac{100}{n^2\pi^2}\left(-\sin\frac{n\pi}{2}\right)\right]$$

$$b_n = \frac{1}{5}\cdot\frac{200}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{40}{n^2\pi^2}\sin\frac{n\pi}{2}$$

The required solution is obtained by substituting this value of b_n in (2).

Thus $$u(x, t) = \frac{40}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\, e^{\frac{-n^2\pi^2c^2t}{100}}\sin\frac{n\pi x}{10}$$

---

15. *Solve the heat equation* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ *with boundary conditions* $u(0, t) = 0 = u(1, t)$ *and* $u(x, 0) = 3\sin\pi x$ *where* $0 < x < 1,\ t > 0$

>> Comparing the given p.d.e with that of the standard form : $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$,

we have $c^2 = 1$

The associated befitting form of the solution is represented by

$$u(x, t) = e^{-p^2 t}(A \cos px + B \sin px) \qquad \ldots (1)$$

Consider $u(0, t) = 0$. Now (1) becomes

$$0 = e^{-p^2 t}(A) \qquad \therefore \; A = 0$$

Consider $u(1, t) = 0$. Using $A = 0$, (1) becomes

$$0 = e^{-p^2 t}(B \sin p)$$

Since $B \neq 0$, $\sin p = 0$ or $p = n\pi$

Now $u(x, t) = e^{-n^2\pi^2 t}(B \sin n\pi x)$

In general, $u(x, t) = \sum_{n=1}^{\infty} b_n \, e^{-n^2\pi^2 t} \sin n\pi x \qquad \ldots (2)$

Consider $u(x, 0) = 3 \sin \pi x$ and we have from (2),

$$3 \sin \pi x = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

ie., $\quad 3 \sin \pi x = b_1 \sin \pi x + b_2 \sin 2\pi x + b_3 \sin 3\pi x + \cdots$

Comparing both sides we get $b_1 = 3, \; b_2 = 0, \; b_3 = 0, \; \cdots$

We substitute these values in the expanded form of (2).

Thus $\quad \mathbf{u(x, t) = 3 e^{-\pi^2 t} \sin \pi x}$

---

16. *Show that the solution of the B.V.P :* $u_t = c^2 u_{xx}$; $u(0, t) = 0 = u(\pi, t)$ *and* $u = \pi x - x^2$ *when* $t = 0$ *in* $(0, \pi)$ *is given by*

$$\frac{8}{\pi}\left[\frac{1}{1^3} e^{-c^2 t} \sin x + \frac{1}{3^3} e^{-9c^2 t} \sin 3x + \frac{1}{5^3} e^{-25c^2 t} \sin 5x + \cdots\right]$$

>> The befitting solution for solving the given B.V.P is represented by

$$u(x, t) = e^{-c^2 p^2 t}(A \cos px + B \sin px) \qquad \ldots (1)$$

Consider $u(0, t) = 0$. Now (1) becomes

$$0 = e^{-c^2 p^2 t}(A) \qquad \therefore \; A = 0$$

Consider $u(\pi, t) = 0$. Using $A = 0$ (1) becomes

$$0 = e^{-c^2 p^2 t}(B \sin p\pi)$$

Since $B \neq 0$, $\sin p\pi = 0 \quad \therefore \; p = n$

Now $u(x,t) = e^{-c^2 n^2 t}\,(B \sin nx)$

In general, $u(x,t) = \sum_{n=1}^{\infty} b_n\, e^{-c^2 n^2 t} \sin nx$ ... (2)

$u(x,0) = \pi x - x^2$ and we have from (2)

$$\pi x - x^2 = \sum_{n=1}^{\infty} b_n \sin nx$$

The series in R.H.S is regarded as the sine half range Fourier series of $(\pi x - x^2)$ in $(0, \pi)$ and hence

$$b_n = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin nx \, dx$$

$$b_n = \frac{2}{\pi}\left[(\pi x - x^2)\cdot\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\cdot\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$b_n = \frac{-4}{\pi n^3}[\cos nx]_0^{\pi} = \frac{-4}{\pi n^3}\{\cos n\pi - 1\} = \frac{4}{\pi n^3}\{1 - (-1)^n\}$$

$\therefore$ $b_n = \dfrac{8}{\pi n^3}$ when n is odd and $b_n = 0$ when n is even.

Hence we have from (2),

$$u(x,t) = \sum_{n=1,3,5\cdots}^{\infty} \frac{8}{\pi n^3}\, e^{-c^2 n^2 t} \sin nx$$

Thus $u(x,t) = \dfrac{8}{\pi}\left[\dfrac{1}{1^3}\, e^{-c^2 t} \sin x + \dfrac{1}{3^3}\, e^{-9c^2 t} \sin 3x + \dfrac{1}{5^3}\, e^{-25 c^2 t} \sin 5x + \cdots\right]$

---

17. *Solve the B.V.P :* $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$, $0 < x < l$;

$$\frac{\partial u}{\partial x}(0,t) = 0,\quad \frac{\partial u}{\partial x}(l,t) = 0,\quad u(x,0) = x$$

>> The befitting solution for solving the given B.V.P is represented by

$u(x,t) = e^{-c^2 p^2 t}\,(A \cos px + B \sin px)$ ... (1)

$$\frac{\partial u}{\partial x} = e^{-c^2 p^2 t}\,(-pA \sin px + pB \cos px)$$

$$\frac{\partial u}{\partial x}(0, t) = e^{-c^2p^2t}(pB) = 0 \quad \therefore \ B = 0$$

Also $$\frac{\partial u}{\partial x}(l, t) = e^{-c^2p^2t}(-pA \sin pl) = 0$$

Since $B = 0$, A cannot be zero and hence we must have,

$$\sin pl = 0 \text{ or } pl = n\pi \quad \therefore \ p = n\pi/l$$

Now $$u(x, t) = e^{\frac{-n^2\pi^2c^2t}{l^2}}\left(A \cos \frac{n\pi x}{l}\right)$$

In general $$u(x, t) = \sum_{n=0}^{\infty} a_n e^{\frac{-n^2\pi^2c^2t}{l^2}} \cos \frac{n\pi x}{l} \qquad \ldots (2)$$

Here we have taken $n = 0, 1, 2, 3, \cdots$ inview of the term $\cos(n\pi x/l)$ in the summation.

Consider $u(x, 0) = x$ and we have from (2)

$$x = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{l}$$

ie., $$x = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

or $$x = \frac{A_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}, \quad \text{where } a_0 = \frac{A_0}{2}$$

The series in R.H.S is regarded as the cosine half range Fourier series of x in $(0, l)$ and hence

$$A_0 = \frac{2}{l}\int_0^l x \, dx \quad \text{and} \quad a_n = \frac{2}{l}\int_0^l x \cos \frac{n\pi x}{l} \, dx$$

Now, $$A_0 = \frac{2}{l}\left[\frac{x^2}{2}\right]_0^l = l \quad \therefore \quad a_0 = \frac{A_0}{2} = \frac{l}{2}$$

Also $a_n = \frac{2}{l} \int_0^l x \cos \frac{n\pi x}{l}\, dx$

$$= \frac{2}{l} \left[x \cdot \frac{\sin \frac{n\pi x}{l}}{(n\pi/l)} - 1 \cdot \frac{-\cos \frac{n\pi x}{l}}{(n\pi/l)^2} \right]_0^l$$

$$a_n = \frac{2}{l} \cdot \frac{l^2}{n^2\pi^2} \left[\cos \frac{n\pi x}{l} \right]_0^l = 2 \frac{l}{n^2\pi^2} (\cos n\pi - 1) = \frac{2l}{n^2\pi^2} \{ (-1)^n - 1 \}$$

$\therefore \quad a_n = \frac{-4l}{n^2\pi^2}$ when n is odd and $a_n = 0$ when n is even.

We shall write (2) in the form,

$$u(x, t) = a_0 + \sum_{n=1}^{\infty} a_n e^{\frac{-n^2\pi^2c^2t}{l^2}} \cos \frac{n\pi x}{l}$$

Thus $\mathbf{u(x, t) = \frac{l}{2} + \sum_{n=1,3,5,\ldots}^{\infty} \frac{-4l}{n^2\pi^2} e^{\frac{-n^2\pi^2c^2t}{l^2}} \cos \frac{n\pi x}{l}}$

---

18. *Solve the heat equation* $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ *subject to the conditions : u is not infinite when* $t \to \infty$, $\frac{\partial u}{\partial x} = 0$ *for* $x = 0$ *and* $x = l$, $u = lx - x^2$ *for* $t = 0$ *between* $x = 0$ *and* $x = l$

>> [*The conditions are of the same type as in problem-17*]

We note that in the befitting solution $e^{-c^2p^2t} \to 0$ as $t \to \infty$. The first of the given condition is satisfied.

[***We need to retrace all the steps of problem-17. We have to compute*** a_0 ***and*** a_n ***by taking*** $(lx - x^2)$]

$$A_0 = \frac{2}{l} \int_0^l (lx - x^2)\, dx$$

$$A_0 = \frac{2}{l} \left[\frac{lx^2}{2} - \frac{x^3}{3} \right]_0^l = \frac{2}{l} \left(\frac{l^3}{2} - \frac{l^3}{3} \right) = \frac{l^2}{3}$$

$\therefore \quad a_0 = A_0/2 = l^2/6$

$$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\,dx$$

$$a_n = \frac{2}{l}\left[(lx-x^2)\cdot\frac{\sin\frac{n\pi x}{l}}{(n\pi/l)} - (l-2x)\cdot\frac{-\cos\frac{n\pi x}{l}}{(n\pi/l)^2} + (-2)\cdot\frac{-\sin\frac{n\pi x}{l}}{(n\pi/l)^3}\right]_0^l$$

$$= \frac{2}{l}\cdot\frac{l^2}{n^2\pi^2}\left[(l-2x)\cos\frac{n\pi x}{l}\right]_0^l$$

$$a_n = 2\frac{l}{n^2\pi^2}\{-l\cos n\pi - l\} = \frac{-2l^2}{n^2\pi^2}\{\cos n\pi + 1\} = \frac{-2l^2}{n^2\pi^2}\{(-1)^n + 1\}$$

$\therefore \quad a_n = \dfrac{-4l^2}{n^2\pi^2}$, when $n = 2, 4, \cdots$ and $a_n = 0$, when $n = 1, 3, \cdots$

Thus $$\boldsymbol{u(x,\ t) = \frac{l^2}{6} + \sum_{n=2,4,6\cdots}^{\infty} \frac{-4l^2}{n^2\pi^2}\, e^{\frac{-n^2\pi^2c^2t}{l^2}} \cos\frac{n\pi x}{l}}$$

---

19. *Find the solution of* $\dfrac{\partial\theta}{\partial x} = k\dfrac{\partial^2\theta}{\partial x^2}$ *such that*

(i) θ *is not infinite when* $t \to +\infty$

(ii) $\dfrac{\partial\theta}{\partial x} = 0$ *when* $x = 0$, $\theta = 0$ *when* $x = l$ *for all values of* t

(iii) $\theta = \theta_0$, *when* $t = 0$ *for all values of* x *in* $(0, l)$

>> Comparing the given p.d.e with that of the standard form : $\dfrac{\partial u}{\partial t} = c^2\dfrac{\partial^2 u}{\partial x^2}$, we have $c^2 = k$ and $u = \theta$

The associated befitting form of solution is represented by

$$\theta(x,\ t) = e^{-kp^2t}(A\cos px + B\sin px) \qquad (1)$$

As $\quad t \to +\infty,\ e^{-kp^2t} \to 0$ (*when* $k > 0$)

$\theta(x,\ t) \to 0$, the first condition is satisfied

From (1), $\frac{\partial\theta}{\partial x} = e^{-kp^2t}(-pA \sin px + pB \cos px)$

Consider $\frac{\partial\theta}{\partial x} = 0$ when $x = 0$.

We now have, $0 = e^{-kp^2t}(pB) \quad \therefore B = 0$

Consider $\theta = 0$ when $x = l$. Using $B = 0$ (1) becomes

$$0 = e^{-kp^2t}(A \cos pl)$$

Since $B = 0$, A cannot be zero and we must have,

$\cos pl = 0$ or pl is an odd multiple of $\pi/2$

ie., $\quad pl = (2n-1)\frac{\pi}{2} \quad \therefore p = (2n-1)\frac{\pi}{2l}; \; n = 1, 2, 3, \cdots$

Now from (1), $\theta(x, t) = e^{-(2n-1)^2(\pi/2l)^2 kt} A \cos\frac{(2n-1)\pi x}{2l}$

In general, $\theta(x, t) = \sum_{n=1}^{\infty} a_n e^{-(2n-1)^2(\pi/2l)^2 kt} \cos\frac{(2n-1)\pi x}{2l} \quad \ldots (2)$

Consider $\theta = \theta_0$ when $t = 0$. We have from (2),

$$\theta_0 = \sum_{n=1}^{\infty} a_n \cos\frac{(2n-1)\pi x}{2l}$$

The series in R.H.S is regarded as the cosine half range Fourier series of θ_0 in $(0, l)$ and hence

$$a_n = \frac{2}{l}\int_0^l \theta_0 \cos\frac{(2n-1)\pi x}{2l}\, dx$$

$$= \frac{2\theta_0}{l}\left[\sin\frac{(2n-1)\pi x}{2l} \Big/ \frac{(2n-1)\pi}{2l}\right]_0^l$$

$$a_n = \frac{2\theta_0}{l}\cdot\frac{2l}{(2n-1)\pi}\cdot\sin(2n-1)\frac{\pi}{2} = \frac{4\theta_0}{\pi(2n-1)}\sin(2n-1)\frac{\pi}{2}$$

We substitute this value of a_n in (2). $\theta(x, t)$ is given by

$$\frac{4\theta_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)\frac{\pi}{2} e^{-(2n-1)^2 (\pi/2l)^2 kt} \cos\frac{(2n-1)\pi x}{2l}$$

Thus $\theta(x, t) = \dfrac{4\theta_0}{\pi}\left[e^{-(\pi/2l)^2 kt} \cos\dfrac{\pi x}{2l} - \dfrac{1}{3} e^{-(3\pi/2l)^2 kt} \cos\dfrac{3\pi x}{2l} + \cdots \right]$

Problems on two dimensional Laplace's equation

Note : The various possible solutions of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ are

(i) $u = (c_1 x + c_2)(c_3 y + c_4)$

(ii) $u = (c_1' e^{px} + c_2' e^{-px})(c_3' \cos py + c_4' \sin py)$

(iii) $u = (c_1'' \cos px + c_2'' \sin px)(c_3'' e^{py} + c_4'' e^{-py})$

We take note of all the given conditions and choose the befitting solution. Usually (iii) will be befitting as it involves periodic functions in the independent variable x.

---

20. *Solve* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ *subject to the conditons.*

$u(0, y) = 0$, $u(\pi, y) = 0$, $u(x, \infty) = 0$ *and* $u(x, 0) = k \sin 2x$

>> The befitting solution to solve the given problem is represented by

$$u(x, y) = (A \cos px + B \sin px)(C e^{py} + D e^{-py}) \quad \ldots (1)$$

$u(0, y) = 0$ gives $(A)(Ce^{py} + De^{-py}) = 0 \quad \therefore A = 0$

$u(\pi, y) = 0$ gives $(B \sin p\pi)(Ce^{py} + De^{-py}) = 0$

Since $A = 0$, $B \neq 0$ and we must have

$\sin p\pi = 0 \quad \therefore \quad p = n$ where n is an integer.

Now $u(x, y) = (B \sin nx)(Ce^{ny} + De^{-ny})$

The condition $u(x, \infty) = 0$ means that $u \to 0$ as $y \to \infty$

ie., $0 = (B \sin nx)(Ce^{ny})$ since $e^{-ny} \to 0$ as $y \to \infty$

Since $B \neq 0$ we must have $C = 0$

We now have, $u(x, y) = BD \sin nx \, e^{-ny}$

Taking $n = 1, 2, 3, \cdots$ and $BD = b_1, b_2, b_3 \cdots$

we obtain a set of independent solutions satisfying the first three conditions. Their sum also satisfy these conditions. Hence we write

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin nx \; e^{-ny} \qquad \ldots (2)$$

Consider $u(x, 0) = k \sin 2x$ and we have from (2),

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx$$

ie., $\quad k \sin 2x = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$

Comparing both sides we get, $b_1 = 0,\ b_2 = k,\ b_3 = 0,\ b_4 = 0 \cdots$

Thus by substituting these values in the expanded form of (2) we have the required solution,

$$\boldsymbol{u(x, y) = k \sin 2x \; e^{-2y}}$$

---

21. *Solve Laplace's equation* $u_{xx} + u_{yy} = 0$ *subject to the conditions* $u(0, y) = u(l, y) = u(x, 0) = 0$ *and* $u(x, a) = \sin(\pi x/l)$

>> The befitting solution to solve the given problem is represented by

$$u(x, y) = (A \cos px + B \sin px)\,(C e^{py} + D e^{-py}) \qquad \ldots (1)$$

$$u(0, y) = 0 \text{ gives } (A)\,(C e^{py} + D e^{-py}) = 0 \qquad \therefore\ A = 0$$

$$u(l, y) = 0 \text{ gives } (B \sin pl)\,(C e^{py} + D e^{-py}) = 0$$

Since $A = 0$, B cannot be zero and we must have

$$\sin pl = 0 \text{ or } pl = n\pi \qquad \therefore\ p = n\pi/l$$

Now $\quad u(x, y) = \left(B \sin \frac{n\pi x}{l}\right)\left(C e^{\frac{n\pi y}{l}} + D e^{-\frac{n\pi y}{l}}\right)$

$$u(x, 0) = 0 \text{ gives } B \sin \frac{n\pi x}{l}\,(C + D) = 0$$

Since $B \neq 0$, $C + D = 0$ or $D = -C$

We can now write, $u(x, y) = BC \sin \frac{n\pi x}{l}\left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right)$

ie., $\quad u(x, y) = 2\,BC \sin \frac{n\pi x}{l} \sinh \frac{n\pi y}{l}$

Putting $n = 1, 2, 3, \cdots$ and taking $2BC = b_1, b_2, b_3 \cdots$ respectively and adding we have,

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \sinh \frac{n\pi y}{l} \qquad \ldots (2)$$

Finally we consider $u(x, a) = \sin(\pi x/l)$

Now, $u(x, a) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \sinh \frac{n\pi a}{l}$

ie., $\sin \frac{\pi x}{l} = b_1 \sin \frac{\pi x}{l} \sinh \frac{\pi a}{l} + b_2 \sin \frac{2\pi x}{l} \sinh \frac{2\pi a}{l} + \cdots$

Comparing both sides we have,

$$b_1 \sinh(\pi a/l) = 1 \quad \therefore \quad b_1 = \frac{1}{\sinh(\pi a/l)},\ b_2 = 0,\ b_3 = 0, \ldots$$

We substitute these values in the expanded form of (2).

Thus $$\boldsymbol{u(x, y) = \frac{\sin \frac{\pi x}{l} \sinh \frac{\pi y}{l}}{\sinh(\pi a/l)}}$$

---

22. *Solve* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *given that* $u(0, y) = 0 = u(l, y) = u(x, 0)$ *and* $u(x, l) = lx - x^2$

>> ***[This problem is similar to problem - 21. We need to retrace all the steps upto the stage of getting equation (2)]***

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \sinh \frac{n\pi y}{l} \qquad \ldots (2)$$

Consider $u(x, l) = lx - x^2$

Now $u(x, l) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \sinh n\pi$

i.e., $lx - x^2 = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$ where $B_n = b_n \sinh n\pi$

The series in R.H.S is regarded as the sine half range Fourier series of $(lx - x^2)$ in $(0, l)$ and hence we have,

$$B_n = \frac{2}{l} \int_0^l (lx - x^2) \sin \frac{n\pi x}{l}\, dx$$

[Refer problem-33 of unit - I for the integration process]

$$B_n = \frac{4\,l^2}{n^3\,\pi^3}\;\;1-(-1)^n$$

or $$B_n = \frac{8\,l^2}{n^3\,\pi^3} \text{ if } n = 1,3,5,\ldots \text{ and } B_n = 0 \text{ if } n = 2,4,6,\ldots$$

i.e., $$b_n \sin h\, n\,\pi = \frac{8\,l^2}{n^3\,\pi^3} \text{ when } n = 1,3,5,\ldots$$

$$\therefore \quad b_n = \frac{8\,l^2}{\sin h\, n\,\pi \cdot n^3\,\pi^3} \text{ where } n = 1,3,5,\ldots$$

We substitute this value of b_n in (2).

Thus $$u(x,y) = \frac{8l^2}{\pi^3} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3 \sin h\, n\,\pi} \sin\frac{n\,\pi\,x}{l} \sin h\,\frac{n\,\pi\,y}{l}$$

3.6 D' Alembert's solution of the one dimensional wave equation

We have one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2} \qquad \ldots (1)$$

[*In this method, the solution $u = u(x,t)$ is found by introducing two new variables v and w interms of t thereby making $u(x,t)$ a composite function. In other words u is considered a function of v and w where v and w are functions of x, t for computing the partial derivatives by applying the chain rule*]

Let $v = x+ct$ and $w = x-ct$

We treat u as a function of v and w which are functions of x and t.

By chain rule we have,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial x}$$

Since $v = x+ct$ and $w = x-ct$, $\dfrac{\partial v}{\partial x} = 1$ and $\dfrac{\partial w}{\partial x} = 1$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\cdot(1) + \frac{\partial u}{\partial w}\cdot(1) = \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}$$

Now, $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right)$$

Again by applying the chain rule we have,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial v}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right) \cdot \frac{\partial v}{\partial x} + \frac{\partial}{\partial w}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right) \cdot \frac{\partial w}{\partial x}$$

ie., $$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2 u}{\partial v^2} + \frac{\partial^2 u}{\partial v\,\partial w}\right) \cdot (1) + \left(\frac{\partial^2 u}{\partial w\,\partial v} + \frac{\partial^2 u}{\partial w^2}\right) \cdot (1)$$

But $$\frac{\partial^2 u}{\partial v\,\partial w} = \frac{\partial^2 u}{\partial w\,\partial v}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial v^2} + 2\,\frac{\partial^2 u}{\partial w\,\partial v} + \frac{\partial^2 u}{\partial w^2} \qquad \ldots (2)$$

Similarly, $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial t} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial t}$$

Since $v = x + ct$ and $w = x - ct$, $\frac{\partial v}{\partial t} = c$ and $\frac{\partial w}{\partial t} = -c$

$$\therefore \quad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial v} \cdot (c) + \frac{\partial u}{\partial w} \cdot (-c) \quad \text{or} \quad \frac{\partial u}{\partial t} = c\left(\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right)$$

Again by applying the chain rule we have,

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial v}\left[c\left(\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right)\right] \cdot \frac{\partial v}{\partial t} + \frac{\partial}{\partial w}\left[c\left(\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right)\right] \cdot \frac{\partial w}{\partial t}$$

ie., $$\frac{\partial^2 u}{\partial t^2} = c\left[\frac{\partial^2 u}{\partial v^2} - \frac{\partial^2 u}{\partial v\,\partial w}\right] \cdot (c) + c\left[\frac{\partial^2 u}{\partial w\,\partial v} - \frac{\partial^2 u}{\partial w^2}\right] \cdot (-c)$$

ie., $$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial v^2} - \frac{\partial^2 u}{\partial v\,\partial w}\right) - c^2\left(\frac{\partial^2 u}{\partial w\,\partial v} - \frac{\partial^2 u}{\partial w^2}\right)$$

$$\therefore \quad \frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial v^2} - 2\,\frac{\partial^2 u}{\partial w\,\partial v} + \frac{\partial^2 u}{\partial w^2}\right) \qquad \ldots (3)$$

Substituting (2) and (3) in (1) we have,

$$c^2\left[\frac{\partial^2 u}{\partial v^2} - 2\,\frac{\partial^2 u}{\partial w\,\partial v} + \frac{\partial^2 u}{\partial w^2}\right] = c^2\left[\frac{\partial^2 u}{\partial v^2} + 2\,\frac{\partial^2 u}{\partial w\,\partial v} + \frac{\partial^2 u}{\partial w^2}\right]$$

ie., $$-4\,\frac{\partial^2 u}{\partial w\,\partial v} = 0 \quad \text{or} \quad \frac{\partial^2 u}{\partial w\,\partial v} = 0$$

We solve this PDE by direct integration, writing it in the form,

$$\frac{\partial}{\partial w}\left(\frac{\partial u}{\partial v}\right) = 0$$

Integrating w.r.t w treating v as constant, we get, $\frac{\partial u}{\partial v} = f(v)$

Now integrating w.r.t v, we get, $u = \int f(v)\, dv + G(w)$

ie., $u = F(v) + G(w)$ where $F(v) = \int f(v)\, dv$

But $v = x + ct$ and $w = x - ct$

Thus $\boldsymbol{u = u(x, t) = F(x+ct) + G(x-ct)}$

This is the **D' Alembert's solution** of the one dimensional wave equation.

WORKED PROBLEMS

23. *Obtain the D'Almbert's solution of the wave equation* $u_{tt} = c^2 u_{xx}$ *subject to the conditions* $u(x, 0) = f(x)$ *and* $\frac{\partial u}{\partial t}(x, 0) = 0.$

>> The D' Alembert's solution of the wave equation is given by

$$u(x, t) = F(x+ct) + G(x-ct) \quad \ldots (1)$$

Consider $u(x, 0) = f(x)$. Now (1) becomes

$$u(x, 0) = F(x) + G(x)$$

ie., $f(x) = F(x) + G(x)$ ... (2)

Differentiating (i) partially w.r.t t we have,

$$\frac{\partial u}{\partial t}(x, t) = F'(x+ct)\cdot(c) + G'(x-ct)\cdot(-c)$$

Now $\frac{\partial u}{\partial t}(x, 0) = c[F'(x) - G'(x)]$

ie., $0 = c[F'(x) - G'(x)]$

or $F'(x) - G'(x) = 0$. Integrating w.r.t x we have,

$$F(x) - G(x) = k \quad \ldots (3)$$

where k is the constant of integration.

By solving simultaneously the equations,

$$F(x)+G(x)=f(x) \quad \ldots(2)$$

$$F(x)-G(x)=k \quad \ldots(3)$$

we obtain, $F(x)=\frac{1}{2}[f(x)+k]$ and $G(x)=\frac{1}{2}[f(x)-k]$

$\therefore \quad F(x+ct)=\frac{1}{2}[f(x+ct)+k]$ and $G(x-ct)=\frac{1}{2}[f(x-ct)-k]$

Substituting these in (1) we have,

$$u(x,t)=\frac{1}{2}[f(x+ct)+k]+\frac{1}{2}[f(x-ct)-k]$$

Thus the required solution is given by

$$\boldsymbol{u(x,t)=\frac{1}{2}\left[f(x+ct)+f(x-ct)\right]}$$

---

24. *Obtian the D' Alembert's solution of the one dimensional wave equation* $u_{tt}=c^2 u_{xx}$ *given that* $u(x,0)=f(x)=l^2-x^2$ *and* $u_t(x,0)=0$

>> [*We can assume the D' Alembert's solution and then retrace all the steps of problem-23*]

$$u(x,t)=\frac{1}{2}\left[f(x+ct)+f(x-ct)\right] \text{ and } f(x)=l^2-x^2, \text{by data.}$$

$$\therefore \quad u(x,t)=\frac{1}{2}\left[l^2-(x+ct)^2+l^2-(x-ct)^2\right]$$

$$=\frac{1}{2}[2l^2-2x^2-2c^2t^2]$$

Thus $\boldsymbol{u(x,t)=l^2-x^2-c^2t^2}$

---

25. *Find the D' Alembert's solution of the wave equation* $u_{tt}=c^2 u_{xx}$ *subject to the conditions,* $u(x,0)=a\sin^2\pi x$ *and* $\frac{\partial u}{\partial t}=0$ *when* $t=0$

>> [*We can assume the D' Alembert's solution and then retrace all the steps of problem-23*]

$$u(x,t)=\frac{1}{2}\left[f(x+ct)+f(x-ct)\right]$$

By data $f(x)=a\sin^2\pi x$ or $f(x)=\frac{a}{2}(1-\cos 2\pi x)$

$$\therefore \quad u(x,t) = \frac{1}{2}\cdot\frac{a}{2}\left[\{1-\cos 2\pi(x+ct)\} + \{1-\cos 2\pi(x-ct)\}\right]$$

$$= \frac{a}{4}\left[2-\{\cos(2\pi x+2\pi ct)+\cos(2\pi x-2\pi ct)\}\right]$$

$$= \frac{a}{4}\left[2-2\cos 2\pi x\cdot\cos 2\pi ct\right]$$

Thus $\boldsymbol{u(x,t) = \frac{a}{2}\left[1-\cos 2\pi x\cdot\cos 2\pi ct\right]}$

EXERCISES

1. Solve the wave equation $u_{tt} = c^2 u_{xx}$ under the conditions : $u(0,t) = 0 = u(l,t)$, $\frac{\partial u}{\partial t}(x,0) = 0$ and $u(x,0) = a\sin(\pi x/l)\cos(5\pi x/l)$

2. Solve the wave equation $u_{tt} = 4u_{xx}$ subject to the conditions : $u(0,t) = 0 = u(\pi,t)$, $u_t(x,0) = 0$ and $u(x,0) = f(x)$ where,

$$f(x) = \begin{cases} x & in\ 0 \le x \le \pi/2 \\ \pi - x & in\ \pi/2 \le x \le \pi \end{cases}$$

3. Solve the wave equation $u_{tt} = c^2 u_{xx}$ given that $u(0,t) = 0 = u(l,t)$, $u(x,0) = 0$ and $u_t(x,0) = \sin^3(\pi x/l)$

4. Given the wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ and the conditions $u(x,0) = x(l-x)$ and $\frac{\partial u}{\partial t}(x,0) = 0$, show that the D'Alembert's solution of the problem is $u(x,t) = lx - x^2 - t^2$

5. Solve the heat equation $u_t = c^2 u_{xx}$ given that $u(0,t) = 0$, $u(\pi,t) = 0$ and $u(x,0) = x(\pi^2 - x^2)$

6. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with the boundary conditions $u(0,t) = 0$, $u(l,t) = 0$ and $u(x,0) = 3\sin(n\pi x)$

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7. Solve the B.V.P. : $u_t = c^2 u_{xx}$, $0 \le x \le 5$, $\frac{\partial u}{\partial x}(0, t) = 0$, $\frac{\partial u}{\partial x}(5, t) = 0$ and $u(x, 0) = x$

8. Solve the heat equation $u_t = c^2 u_{xx}$ given that $u(0, t) = 0$, $u(40, t) = 0$ and $u(x, 0) = 2x + 20$

9. Solve the Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the conditions. $u(0, y) = 0$, $u(\pi, y) = 0$, $u(x, \infty) = 0$ and $u(x, 0) = u_0$ where $0 < x < \pi$, $0 < y < \pi$

10. Solve $u_{xx} + u_{yy} = 0$ given that $u(0, y) = 0$, $u(10, y) = 0$, $u(x, 0) = 0$ and $u(x, 10) = f(x)$

where $f(x) = \begin{cases} x & \text{in } 0 \le x \le 5 \\ 10 - x & \text{in } 5 \le x \le 10 \end{cases}$

ANSWERS

1. $u(x, t) = \frac{a}{2}\left[\sin\frac{6\pi x}{l}\cos\frac{6\pi ct}{l} - \sin\frac{4\pi x}{l}\cos\frac{4\pi ct}{l}\right]$

2. $u(x, t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin nx\cos 2nt$

3. $u(x, t) = \frac{3l}{4c\pi}\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l} - \frac{1}{12c\pi}\sin\frac{3\pi x}{l}\sin\frac{3\pi ct}{l}$

5. $u(x, t) = 12\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}e^{-n^2c^2t}\sin nx$

6. $u(x, t) = 3\sum_{n=1}^{\infty}e^{-n^2\pi^2 t}\sin(n\pi x)$

7. $u(x, t) = \frac{5}{2} + \frac{10}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]e^{-n^2\pi^2c^2t/25}\cos\frac{n\pi x}{5}$

8. $u(x, t) = \sum_{n=1}^{\infty}\frac{40}{n\pi}\left\{1 - 5(-1)^n\right\}e^{-n^2\pi^2c^2t/1600}\sin\frac{n\pi x}{40}$

9. $u(x, y) = \frac{4u_0}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n}\sin nx\, e^{-y}$

10. $u(x, y) = \frac{40}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{\sin h\, n\pi}\sin\frac{n\pi x}{10}\sin h\frac{n\pi y}{10}$
